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PhD dissertation

Periodic Solutions of Nonlinear Damped Wave Equations on the Whole Euclidean Space

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Introduction

Problem statement and background

In this work, we present some criteria for the existence of periodic solutions of the following damped wave equation in the whole Euclidean space $\mathbb{R}^N$:

$$u_{tt} + \beta u_t = \Delta u - V(x)u + f(t,x,u), \quad t > 0, \; x \in \mathbb{R}^N \tag{1}$$

where $\beta > 0$ is a damping coefficient, Δ is the Laplace operator (with respect to x), $V : \mathbb{R}^N \to \mathbb{R}$ is the potential and $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a time T-periodic function. If f is time-independent, the equation (1) has no non-constant periodic solutions because there exists a functional (called the Lyapunov function or energy functional) that decreases along non-constant solutions. Indeed, for $u \in H^1(\mathbb{R}^N)$ and $v \in L^2(\mathbb{R}^N)$ we define

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} |v(x)|^2 + |\nabla u(x)|^2 + V(x)|u(x)|^2 \, dx - \int_{\mathbb{R}^N} g(x,u(x)) \, dx$$

where $g(x,s) = \int_0^s f(x,\tau)\, d\tau$, and, for a non-constant solution (u,u_t) of (1), one has

$$\frac{d}{dt}\mathcal{E}(u,u_t) = -\beta \int_{\mathbb{R}^N} |u_t(t,x)|^2 \, dx < 0.$$

However, T-periodic solutions may appear if f is a T-periodic time-dependent map.

The main contribution of our work is the analysis of the hyperbolic problem in the whole domain, a topic not widely recognized in the literature. Nevertheless, there are numerous articles related to the existence of periodic solutions for hyperbolic equations in bounded domains using various techniques.

One of the methods for finding periodic solutions used in this work is based on the *translation along trajectories operator* (also called the *Poincaré operator*), denoted by $\Phi_T : \mathbb{X} \to \mathbb{X}$. This operator is defined on the space $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ as $\Phi_T(\overline{u},\overline{v}) = (u(T), u_t(T))$, where (u,u_t) is a solution of (1) with initial conditions $(u(0),u_t(0)) = (\overline{u},\overline{v})$. Clearly, the fixed points of Φ_T determine T-periodic solutions of (1). The method of the translation along trajectories operator has been developed for many years by numerous authors. The earliest works include [11] by F. Browder, who considered evolution equations in a Hilbert space generating an evolution system, and [51] by J. Prüss, who studied semilinear evolution equations with the generator of a compact C_0-semigroup. More recently, P. Kokocki in [39] and [40] proved the existence of periodic solutions for nonlinear evolution equations at resonance involving parabolic equations. The method of the translation operator was also applied to obtain the existence of periodic solutions for hyperbolic equations. For example, A. Ćwiszewski used it in [15] for damped hyperbolic equations at resonance and in [16] to prove that the strongly damped beam equation admits periodic solutions near equilibrium states. Moreover, P. Kokocki obtained the existence of periodic solutions for strongly damped hyperbolic equations at resonance in [41]. The translation along trajectories operator can also be used to characterize the stability of periodic solutions, as shown by R. Ortega in his work [45] on the nonlinear telegraph equation.

Alternatively, one may look for periodic solutions as the solutions of the coincidence problem

$$Lu = N_F(u) \tag{2}$$

where L is the linear operator and N_F is determined by the nonlinear term in the equation, both acting between function spaces that depend on time and spatial variable. S. Fučík and J. Mawhin [29] applied this approach to find periodic solutions to the telegraph equation using the degree theory for Fredholm operators. A variant of this method was used by L. Cesari and R. Kannan [13] to get the existence of periodic solutions of nonlinear wave equations with linear damping when the kernel of the underlying linear operator is infinite dimensional. H. Amann and E. Zehnder [3] applied variational methods and critical points theory to find solutions to abstract equations in a real Hilbert space involving (2) and, among others, found periodic solutions to a semilinear wave equation under a non-resonance condition at infinity [3, Sec. 11]. To solve (2) H. Brezis and L. Nirenberg [10] also develop some general methods for treating equations in Hilbert space using fixed-point theory and the ideas from the monotone operators theory. In particular, they proved that the nonlinear telegraph equation admits periodic solutions under a resonance condition [10, Sec. V.2]. This approach was further developed by J. Berkovits and V. Mustonen [6] who considered semilinear wave equation and found solutions to the coincidence problem (2) as the solutions of $F(u) = y$, constructing an extension of the Leray-Schauder degree for a class of mappings of the monotone type. They treated both non-resonant and resonant cases (see [6, Sec. 6], paragraphs B and C, respectively).

It is noteworthy that, in a proper sense, the translation operator method and the coincidence method may be, to some extent, equivalent, as shown by R. Ortega and A. M. Robles-Pérez [46] for a class of second-order evolution equations, including the forced sine-Gordon equation.

To the best of the author's knowledge, only a few articles address hyperbolic equations in $\mathbb{R}^N$. In a recent paper, Y. Kagei and H. Takeda [37] proved the existence of periodic solutions and the stability result for the system of quasilinear hyperbolic equations with a viscoelastic term in the whole space $\mathbb{R}^3$. The problem of the existence of periodic solutions is related, in the sense that it involves similar challenges, to the question of the existence of global attractors. In this case, there are more results concerning hyperbolic problems in unbounded domains. N. I. Karachalios and N. M. Stavrakakis [38] proved the existence of a global attractor for a semilinear dissipative wave equation in $\mathbb{R}^N$. In [59] S. V. Zelik obtained the existence of a locally compact global attractor for nonlinear dissipative hyperbolic equations in unbounded domains. Recently, P. Ding and Z. Yang [23] showed the existence of *a strong global attractor* and investigated regularity of weak solutions for wave equations in $\mathbb{R}^3$ with a strong damping of the form $(-\Delta)^\alpha u_t$, where $\alpha \in [1/2, 1)$ and nonlinearities of critical growth.

To find the fixed points of the translation operator Φ_T we shall apply the topological fixed-point index. The lack of compact embeddings for the function spaces on $\mathbb{R}^N$ causes that the translation operator Φ_T may not be completely continuous; thus, the Leray-Schauder fixed-point index cannot be applied. However, for some classes of nonlinearities and when properly renorming the space $\mathbb{X}$, Φ_T falls into a class of the *k-set contractions*. To be more precise, we use the spectral properties of the operator $-\Delta + V$ to prove that the translation operators Φ_t, for $t > 0$, are k-set contractions in the linear case, i.e., when $f \equiv 0$. We then use the *tail estimates* method to show that F is either completely continuous or a k-set contraction with a sufficiently small constant. The tail estimates method was originated by B. Wang [57] who studied global attractors for the nonlinear reaction-diffusion equation in $\mathbb{R}^N$. M. Prizzi [49] applied this method to prove the existence of nontrivial equilibria and connecting orbits for parabolic equations in $\mathbb{R}^N$. D. Fall and Y. You [26] used tail estimates to establish the existence of a global attractor for the damped nonlinear wave equation in unbounded domains. Further, M. Prizzi and K. P. Rybakowski [50] used this method to prove the existence of a global attractor for a wider class of damped wave equations in unbounded domains. A. Ćwiszewski and R. Łukasiak applied tail estimates to get the existence of periodic solutions for parabolic problems in $\mathbb{R}^N$, both in resonant [19] and non-resonant case [20].

There are also some other problems arising, such as the averaging principle for equations governed by operators that are not sectorial or the structure of the spectrum of the corresponding C_0-semigroup generator.

To define the operator Φ_T in a rigorous way, we rewrite the damped wave equation (1) as the evolutionary differential equation in the Hilbert space $L^2(\mathbb{R}^N)$:

$$\ddot{u}(t) + \beta\dot{u}(t) = -(-\Delta + \boldsymbol{V})u(t) + F(t, u(t)), \ t > 0$$

where $\Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Laplace operator, $\boldsymbol{V} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the operator of multiplication by V and $F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the *Nemytskii operator* associated with f. The equation above can be formally transformed into a system of first-order differential equations in $\mathbb{X}$:

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F(t, u(t))\big), \quad t > 0 \tag{3}$$

where the so-called *damped wave linear operator* $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is given by

$$\mathbb{A}(u, v) = (v, -(-\Delta + \boldsymbol{V})u - \beta v) \ \text{ for } (u, v) \in D(\mathbb{A}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N).$$

It turns out that $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$ of bounded linear operators, while the map F is Lipschitz continuous, if so is f. Therefore, the classical results on global existence, uniqueness, and continuous dependence on initial conditions for *mild solutions* apply. As a consequence, operator Φ_T is well-defined and continuous. Moreover, we will say that u is a mild solution of (1) if (u, u_t) is a mild solution of (3).

In order to compute the fixed-point index of Φ_T we shall use a combination of several techniques, such as time averaging methods, linearization, and spectral analysis of the operator $\mathbb{A}$ along with the geometric conditions imposed on the nonlinear term f like *Landesman-Lazer* or *sign conditions*. We separately study the *resonant case (at infinity)* when

$$X_0 = \operatorname{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\} \quad \text{and} \quad f \text{ is bounded by a square integrable function}(^1), \tag{4}$$

and the *non-resonant case* when the linear operator $-\Delta + \boldsymbol{V}$ perturbed by a linearization of f at infinity has zero kernel.

In the presence of resonance, the wave equation (1) may not admit T-periodic mild solutions (see [41, Remark 4.1]). However, the Landesman-Lazer or strong resonance conditions are sufficient to guarantee the existence of periodic solutions. The former conditions were introduced by E. M. Landesman and A. C. Lazer [43], who investigated the existence of weak solutions to an elliptic equation with a nonlinear perturbation. The strong resonance conditions were introduced by P. Bartolo, V. Benci, and D. Fortunato in their work [5] on nonlinear problems in bounded domains. J. Mawhin and K. Schmitt [44] examinated an abstract equation at resonance in a Banach space with a linear Fredholm operator and a nonlinearity satisfying a Landesman-Lazer type condition. More recently, A. Fonda and M. Garrione [28] proved that the Landesman-Lazer condition implies the so-called Ahmad-Lazer-Paul condition.

The idea of the resonant averaging principle is to "project" the parametrized equation (3) with F replaced by εF onto $X_0 \times X_0$. One can show that if the asymptotic bottom of the $L^\infty(\mathbb{R}^N)$-part of V is positive, then zero lies in the discrete part of the spectrum of $-\Delta + \boldsymbol{V}$, and, consequently, X_0 is a finite-dimensional subspace. This enables the fixed-point index of the translation operator $\Phi_T^{(\varepsilon)}$ associated with the parametrized equation (3) to be expressed in terms of the Brouwer degree of the time averaged mapping F restricted to X_0, for sufficiently small parameter $\varepsilon \in (0, 1]$. To obtain continuation to the original problem (3), we apply either the Landesman-Lazer or strong resonance conditions. This approach originates in the theory of ODEs (see [33]) and was later extended by A. Schiaffino and K. Schmitt [54] to the evolution equations modeled after parabolic problems. A. Ćwiszewski [15] applied this technique to study the damped wave equations, and P. Kokocki [41] to investigate the strongly damped hyperbolic equations, both in bounded domains.

$(^1)$See condition (f2)$'$ on p. ix.

The averaging principle in the non-resonant case states that the solutions of the family of frequency changing equations (3), in which $F(t, u(t))$ is replaced by $F(t/\varepsilon, u(t))$, converge to the solution of the averaged equation as the frequency increases, i.e., as $\varepsilon \to 0^+$. Moreover, if these solutions are periodic with periods tending to zero, they converge to an equilibrium of the averaged equation. This method was originally developed for ODEs by N. N. Bogoliubov, N. M. Krylov, and Y. A. Mitropolsky (see [7, 8]). The first extension of the averaging principle to infinite-dimensional spaces was introduced by D. Henry [35, Thm. 3.4.9] for evolution equations governed by analytic C_0-semigroups. Later, A. Ćwiszewski and P. Kokocki [17, Prop. 6.4] established the averaging principle for evolution equations where the linear part generates a C_0-semigroup of strict contractions and the nonlinear term is a k-set contraction.

In addition to the averaging principle, we also provide, in the non-resonance case, index formulae for autonomous damped wave equations, along with a *a priori* estimates used to establish a continuation principle. The approach originates from the theory of ODEs, see, for example, M. Furi, M. P. Pera, and M. Spadini [30, Thm. 3.11 and Thm. 4.9], who studied equations on differentiable manifolds embedded in Euclidean spaces. A. Ćwiszewski and P. Kokocki [17, Sec. 6] adopted this technique to evolution equations where the linear operator generates a C_0-semigroup of a strict contraction, and later extended it in [18, Sec. 4] to evolution equations with a family of linear operators generating an evolution system of strict contractions. In both works, the nonlinear term is assumed to be a k-set contraction. Moreover, they proved in [18, Sec. 5] the existence of periodic solutions to hyperbolic equations with a time-dependent damping term, under the assumption that the linearization at infinity does not belong to the spectrum of the linear operator appearing in the equation.

Main results

We begin by stating the precise assumptions on the potential V and the nonlinearity f. Specifically, we assume that V is a *Kato-Rellich type potential*, that is,

$$V = V_\infty + V_0 \quad \text{for} \quad V_\infty \in L^\infty(\mathbb{R}^N),\ V_0 \in L^p(\mathbb{R}^N) \tag{5a}$$

and

$$\begin{cases} p \geq 2 & \text{if } N \in \{1, 2, 3\}, \\ p > 2 & \text{if } N = 4, \\ p \geq N/2 & \text{if } N \geq 5. \end{cases} \tag{5b}$$

The relationship between the exponent p and the dimension N arises from the application of the Kato–Rellich theorem to perturbations of self-adjoint operators, as well as from Sobolev embedding results. This relationship is well established in the literature (see, e.g., [56, Equations (III.4a), (III.4b), (III.4c), p. 3526]). Additionally, we assume that the *asymptotic bottom* of V_∞, defined by

$$\varrho(V_\infty) = \lim_{R \to +\infty} \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty, \tag{6}$$

where $D(0, R) = \{x \in \mathbb{R}^N : |x| \leq R\}$ and $|\cdot|$ denotes the Euclidean norm, is positive:

$$\varrho(V_\infty) > 0. \tag{7}$$

An important example of a Kato-Rellich type potential is the *Coulomb potential* $V_C : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$, given by the formula $V_C(x) = a/|x|$, where $N \geq 3$ and $a \in \mathbb{R}$ (see [36, Example 14.8]). If $V : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is defined as $V(x) = V_C(x) + b$ with $b > 0$, then V also satisfies (7).

We assume that the function $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(C) f is a *Carathéodory function* (satisfies *Carathéodory condition*), that is, for almost every $x \in \mathbb{R}^N$, the map $(t, u) \mapsto f(t, x, u)$ is continuous on $[0, +\infty) \times \mathbb{R}$, and, for all $t \in [0, +\infty)$ and $u \in \mathbb{R}$, the map $x \mapsto f(t, x, u)$ is measurable.

(P) f is *T-periodic in time*, i.e., there exists $T > 0$ such that

$$f(t+T,x,u) = f(t,x,u)$$

for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, $u \in \mathbb{R}$.

In addition, f satisfies the following properties:

(f1) (*Lipschitz condition*) There exists a function $l : \mathbb{R}^N \to \mathbb{R}$ such that

$$|f(t,x,u_1) - f(t,x,u_2)| \leq l(x)|u_1 - u_2|$$

for almost every $x \in \mathbb{R}^N$, all $t \in [0,+\infty)$, u_1, $u_2 \in \mathbb{R}$. The function l is assumed to be of *Kato-Rellich type*, that is,

$$l = l_\infty + l_0 \quad \text{for} \quad l_\infty \in L^\infty(\mathbb{R}^N),\ l_0 \in L^p(\mathbb{R}^N), \tag{8a}$$

where

$$\begin{cases} p \geq 2 & \text{if } N = 1, \\ p > 2 & \text{if } N = 2, \\ p \geq N & \text{if } N \geq 3, \end{cases} \tag{8b}$$

and, in addition, function l satisfies

$$l_\infty(x) \geq 0 \quad \text{and} \quad l_0(x) \geq 0 \quad \text{for almost every } x \in \mathbb{R}^N. \tag{9}$$

(f2) (*Boundedness at zero*) There exists a function $m_0 \in L^2(\mathbb{R}^N)$ such that

$$|f(t,x,0)| \leq m_0(x)$$

for almost every $x \in \mathbb{R}^N$ and all $t \in [0,+\infty)$.

In the resonant case, we assume that condition (4) holds, i.e., $\operatorname{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\}$ and

(f2)′ There exists $m \in L^2(\mathbb{R}^N)$ such that

$$|f(t,x,u)| \leq m(x)$$

for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, $u \in \mathbb{R}$.

Furthermore, we require that one of the following Landesman-Lazer type conditions is satisfied:

$$(LL)'_\pm \quad \begin{cases} \check{h}_+(x) = \displaystyle\int_0^T \check{f}_+(t,x)\,dt \geq 0,\ \hat{h}_-(x) = \int_0^T \hat{f}_-(t,x)\,dt \leq 0 \ \text{ for almost every } x \in \mathbb{R}^N, \\ \check{h}_+\Big|_E > 0 \text{ and } \hat{h}_-\Big|_E < 0 \ \text{ for some } E \subset \mathbb{R}^N \text{ of positive measure} \end{cases}$$

or

$$(LL)'_- \quad \begin{cases} \hat{h}_+(x) = \displaystyle\int_0^T \hat{f}_+(t,x)\,dt \leq 0,\ \check{h}_-(x) = \int_0^T \check{f}_-(t,x)\,dt \geq 0 \ \text{ for almost every } x \in \mathbb{R}^N, \\ \hat{h}_+\Big|_E < 0 \text{ and } \check{h}_-\Big|_E > 0 \ \text{ for some } E \subset \mathbb{R}^N \text{ of positive measure} \end{cases}$$

where

$$\check{f}_\pm(t,x) = \liminf_{u \to \pm\infty} f(t,x,u) \quad \text{and} \quad \hat{f}_\pm(t,x) = \limsup_{u \to \pm\infty} f(t,x,u)$$

for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. Alternatively, we assume that one of the following strong resonance type conditions holds:

$$(SR)'_+ \qquad \begin{cases} \text{There exists } r \in L^2(\mathbb{R}^N) \text{ such that, for almost every } x \in \mathbb{R}^N, \\ r(x) \geq 0 \quad \text{and} \quad uf(t,x,u) \geq 0, \quad \text{for all } |u| \geq r(x) \text{ and all } t \in [0,T], \\ \int_0^T k_\pm(t,x)\,dt > 0 \ \text{ for } x \in E \text{ and some } E \subset \mathbb{R}^N \text{ of positive measure} \end{cases}$$

or

$$(SR)'_- \qquad \begin{cases} \text{There exists } r \in L^2(\mathbb{R}^N) \text{ such that, for almost every } x \in \mathbb{R}^N, \\ r(x) \geq 0 \quad \text{and} \quad uf(t,x,u) \leq 0, \quad \text{for all } |u| \geq r(x) \text{ and all } t \in [0,T], \\ \int_0^T k_\pm(t,x)\,dt < 0 \ \text{ for } x \in E \text{ and some } E \subset \mathbb{R}^N \text{ of positive measure} \end{cases}$$

where the limits

$$k_\pm(t,x) = \lim_{u \to \pm\infty} uf(t,x,u)$$

are assumed to be finite, for almost every $x \in \mathbb{R}^N$ and all $t \in [0,T]$. Based on the *unique continuation property* (see Theorem 4.3.4), one can show that conditions $(LL)'_\pm$ imply the classical Landesman-Lazer conditions (see Lemma 4.3.5) and conditions $(SR)'_\pm$ imply the classical strong resonance conditions (see Lemma 4.3.7).

We state the main result on the existence of periodic solutions for the nonlinear damped wave equation (1) at resonance at infinity.

Theorem I. [an extended version – see Thm. 4.4.1]
Assume that $V : \mathbb{R}^N \to \mathbb{R}$, where $N \geq 3$, is a Kato-Rellich type potential such that the asymptotic bottom of the $L^\infty(\mathbb{R}^N)$-part of V is positive, that is, conditions (5a), (5b), *and* (7) *are satisfied. Suppose that the nonlinear term f is a T-periodic Carathéodory function (see conditions (P) and (C) on p. ix) satisfying the Lipschitz condition* (f1) *(see p. ix). Moreover, assume that the nonlinear damped wave equation* (1) *is at resonance at infinity, i.e.,* $\mathrm{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\}$ *and f satisfies condition* (f2)′ *(see p. ix).*

If one of the conditions $(LL)'_\pm$ or $(SR)'_\pm$ is satisfied, then equation (1) *admits a T-periodic mild solution.*

In the non-resonant case, we assume that f *admits a linearization at infinity* (or it is *asymptotically linear at infinity*), that is, there exists a Carathéodory function $\omega : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that the following condition holds

$$\lim_{|u| \to \infty} \frac{f(t,x,u)}{u} = \omega(t,x) \tag{10}$$

for almost every $x \in \mathbb{R}^N$, uniformly on bounded subsets of $[0,+\infty)$ with respect to t. We then say that the function ω is the *coefficient of the linearization at infinity* (or it is the *asymptotic coefficient at infinity*), and the function $f^\omega : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f^\omega(t,x,u) = \omega(t,x)u$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all $u \in \mathbb{R}$, is the *linearization of f at infinity.* Similarly, we say that f *admits a linearization at zero* (or it is *asymptotically linear at zero*), if there exists a Carathéodory function $\alpha : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{u \to 0} \frac{f(t,x,u)}{u} = \alpha(t,x) \tag{11}$$

for almost every $x \in \mathbb{R}^N$, uniformly on bounded subsets of $[0, +\infty)$ with respect to t. As before, the function α is called the *coefficient of the linearization at zero* (or the *asymptotic coefficient at zero*), and the function $f^\alpha : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f^\alpha(t, x, u) = \alpha(t, x)u$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all $u \in \mathbb{R}$, is called the *linearization of f at zero.*

We define the averaged functions $\widehat{\omega}$, $\widehat{\alpha} : \mathbb{R}^N \to \mathbb{R}$ by

$$\widehat{\omega}(x) = \frac{1}{T}\int_0^T \omega(t, x)\, dt, \quad \widehat{\alpha}(x) = \frac{1}{T}\int_0^T \alpha(t, x)\, dt \quad \text{for almost every } x \in \mathbb{R}^N. \tag{12}$$

In addition, f satisfies condition (f1) with a function l, for which

$$\widehat{\varrho}(l_\infty) < \sqrt{d}\,\rho \tag{13}$$

where

$$\widehat{\varrho}(l_\infty) = \lim_{R \to +\infty} \operatorname*{ess\,sup}_{\mathbb{R}^N \setminus B(0,R)} l_\infty, \tag{14}$$

$d = \operatorname{dist}(0, \sigma(-\Delta + \boldsymbol{V}) \cap (0, +\infty))$ (see (2.54)), and $\rho > 0$ is the strong dissipativity constant of the operator $\mathbb{A}$ (see Proposition 2.3.7).

Now, we state the main result on the existence of periodic solutions for the nonlinear damped wave equation (1) under non-resonance conditions.

Theorem II. [an extended version – see Thm. 5.5.1] *Assume that $V : \mathbb{R}^N \to \mathbb{R}$, where $N \geq 1$, is a Kato-Rellich type potential such that the asymptotic bottom of V_∞ is positive. Suppose that a time T-periodic Carathéodory function $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (f1) *and* (f2)*, and that the function l from* (f1) *satisfies* (13)*.*

Suppose further that f admits both a linearization at infinity and at zero with coefficients denoted by ω, $\alpha : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$, respectively, such that $\widehat{\omega}$, $\widehat{\alpha} : \mathbb{R}^N \to \mathbb{R}$ are Kato-Rellich type functions satisfying

$$\varrho(V_\infty - \widehat{\omega}_\infty) > 0 \quad \text{and} \quad \varrho(V_\infty - \widehat{\alpha}_\infty) > 0,$$

and that the non-resonance conditions both at infinity and at zero hold:

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}) = \{0\} \quad \text{and} \quad \operatorname{Ker}(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}) = \{0\} \tag{15}$$

where $\widehat{\boldsymbol{\omega}}$, $\widehat{\boldsymbol{\alpha}} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ denote the multiplication operators by $\widehat{\omega}$, $\widehat{\alpha}$, respectively, and $-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}$, $-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}$ are regarded as linear operators in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$.

If, for every $\varepsilon \in (0, 1]$, the linear damped wave equations

$$u_{tt} + \beta u_t = \Delta u - V(x)u + \omega(t/\varepsilon, x)u, \quad t > 0,\ x \in \mathbb{R}^N \tag{16}$$

and

$$u_{tt} + \beta u_t = \Delta u - V(x)u + \alpha(t/\varepsilon, x)u, \quad t > 0,\ x \in \mathbb{R}^N \tag{17}$$

admit no nonzero εT-periodic mild solutions, and if

$$m_-(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}) \not\equiv m_-(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}) \pmod 2 \tag{18}$$

where $m_-(\cdot)$ denotes the total multiplicity of negative eigenvalues, then the nonlinear damped wave equation (1) *admits a nonzero T-periodic mild solution.*

Both operators $-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}$ and $-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}$ appear to be self-adjoint. Hence, for each of their eigenvalues, the geometric multiplicity is equal to the algebraic multiplicity (see Proposition 2.1.10 (ii) and Appendix A.1). Moreover, both sets $\sigma(-\Delta+\boldsymbol{V}-\widehat{\boldsymbol{\omega}})\cap(-\infty,0)$ and $\sigma(-\Delta+\boldsymbol{V}-\widehat{\boldsymbol{\alpha}})\cap(-\infty,0)$ consist of finitely many eigenvalues, each with a finite multiplicity (see Proposition 2.1.12 (iii)). Therefore, the numbers $m_-(-\Delta+\boldsymbol{V}-\widehat{\boldsymbol{\omega}})$ and $m_-(-\Delta+\boldsymbol{V}-\widehat{\boldsymbol{\alpha}})$ are well-defined.

In both the resonant and non-resonant cases, we present specific examples of equation (1) to which Theorem I and Theorem II apply. To this end, in each case we consider the Coulomb potential on $\mathbb{R}^3$ and classes of nonlinearities that satisfy the above assumptions.

Outline

In Chapter 1, we briefly recall the basic facts from the theory of C_0-semigroups and evolution equations. Inspired by a result of D. Henry [35, Thm. 3.4.8], we formulate and prove a general continuity theorem for sequences of evolution equations. From this result, we derive a continuity theorem for evolution equations with a parametrized nonlinear term, as well as an averaging principle. We conclude this chapter with a discussion on the existence, continuity, and compactness properties of the translation along trajectories operator.

In Chapter 2, we justify that the operator $-\Delta+\boldsymbol{V}$ is self-adjoint and determine its spectrum by combining the relative compactness of $\boldsymbol{V}_0$, the multiplication operator by V_0, with Persson's formula (see [48, Thm. 2.1] and [36, Sec. 14.4]). We then analyze the spectrum of the damped wave operator $\mathbb{A}$ and show that it generates a C_0-semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$ on $\mathbb{X}$.

Next, we provide the spectral decomposition of $L^2(\mathbb{R}^N)$ with respect to the spectrum of $-\Delta+\boldsymbol{V}$, and subsequently decompose the space $\mathbb{X}$. Using this decomposition, we introduce a scalar product $\langle\cdot,\cdot\rangle_{s,V}$ on $\mathbb{X}$, depending on a parameter $s>0$ and a potential V, which induces a norm $\|\cdot\|_{s,V}$ equivalent to the standard one on $\mathbb{X}$, following the approach of [16, Sec. 2].

Exploiting the geometric structure of $\mathbb{A}$, we show that the semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$ is a k-set contraction with respect to $\chi_{s,V}$ (the Hausdorff measure of non-compactness on $(\mathbb{X},\|\cdot\|_{s,V})$) for a suitably chosen $s>0$. Moreover, we determine the spectrum of the semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$, which allows us to compute the topological fixed-point index for the translation along trajectories operator associated with the linear damped wave equation.

In Chapter 3, we study the Nemytskii operator F associated with a function f. We begin by recalling the basic properties of F and proving some auxiliary results concerning sequences of functions f_n, $n\geq 1$, and sequences of the corresponding Nemytskii operators F_n, $n\geq 1$. Next, we examine the compactness properties in $L^2(\mathbb{R}^N)$ of the set

$$\bigcup_{n=1}^{\infty} F_n(D\times U) \tag{19}$$

where $D\subset[0,+\infty)$ and $U\subset H^1(\mathbb{R}^N)$ are bounded. We show, depending on the assumptions made, that the set (19) is either relatively compact, or its Hausdorff measure of non-compactness can be estimated by $\widehat{\varrho}(l_\infty)\chi_{L^2}(U)$, where $\widehat{\varrho}(l_\infty)$ is defined as in (14) and $\chi_{L^2}(U)$ denotes the Hausdorff measure of non-compactness of the set U in $L^2(\mathbb{R}^N)$. This result implies that the Nemytskii operator F is either completely continuous or a k-set contraction. Furthermore, we show that the translation operator associated with the family of nonlinear damped wave equations of the form (1) is a well-defined, continuous k-set contraction with respect to the measure of non-compactness $\chi_{s,V}$. The chapter concludes by considering a sequence of frequency changing evolution equations and proving that the corresponding sequence of periodic points is relatively compact.

In Chapter 4 we study periodic solutions of the nonlinear damped wave equations at resonance at infinity. We first derive a topological fixed-point index formulae of the translation operator associated with the family of autonomous evolution equations:

$$(\dot u(t),\dot v(t)) = \mathbb{A}(u(t),v(t)) + \varepsilon\big(0,F_0(P_0u(t))\big),\quad t>0,\ \varepsilon\in(0,1],$$

where $F_0 : X_0 \to X_0$ is Lipschitz (the subspace X_0 is defined in (4)), and P_0 is the orthogonal projection in $L^2(\mathbb{R}^N)$ onto X_0. Next, we prove the *resonant averaging principle*, which states that, for sufficiently small $\varepsilon \in (0, 1]$, the topological fixed-point index of the translation operator $\Phi_T^{(\varepsilon)}$, associated with

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon\big(0, F(t, u(t))\big), \quad t > 0,\ \varepsilon \in (0, 1], \tag{20}$$

coincides (up to a sign) with the Brouwer degree $\deg_B(\overline{F}_0, U)$, where $U \subset X_0$ is an open bounded set such that $\overline{F}_0(u) \neq 0$ for $u \in \partial U$ and $\overline{F}_0 : X_0 \to X_0$ is given by

$$\overline{F}_0(u) := \frac{1}{T}\int_0^T P_0 F(t, u)\, dt.$$

As a consequence, we derive a *continuation principle* assuming that $\deg_B(\overline{F}_0, U) \neq 0$ and appropriate *a priori* bounds for periodic solutions of (20) are satisfied. It is well-known that under Landesman-Lazer or strong resonance conditions, the Brouwer degree of $\overline{F}_0$ is nontrivial on a sufficiently large ball. Moreover, these conditions allow us to derive specific geometric conditions that ensure the validity of the bounds required for the continuation principle. In this way, we establish Theorem I. We conclude this chapter by applying Theorem I to the nonlinear damped wave equation (1) with the Coulomb potential on $\mathbb{R}^3$ and appropriate nonlinearities.

Chapter 5 is devoted to the study of periodic solutions of the nonlinear damped wave equations under non-resonance conditions. We begin by establishing an averaging principle, which states that mild solutions of the family of evolution equations

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F(t/\varepsilon, u(t))\big), \quad t > 0,\ \varepsilon \in (0, 1], \tag{21}$$

converge, as $\varepsilon \to 0^+$, to a mild solution of the averaged equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, \widehat{F}(u(t))\big), \quad t > 0 \tag{22}$$

where $\widehat{F} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the time-averaged function $\widehat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. If, in addition, the mild solutions of (21) are εT-periodic, then they converge to a solution of the stationary equation

$$\mathbb{A}(u, v) + (0, \widehat{F}(u)) = 0. \tag{23}$$

We subsequently prove the *index averaging formula*: if an open bounded set $Z \subset \mathbb{X}$ is such that the equation (23) admits no solutions on ∂Z, then, for sufficiently small $\varepsilon \in (0, 1]$, we have

$$\mathrm{Ind}_C(\Psi_{\varepsilon T}^{(\varepsilon)}, Z) = \mathrm{Ind}_C(\widehat{\Psi}_{\varepsilon T}, Z)$$

where $\Psi_t^{(\varepsilon)}$ and $\widehat{\Psi}_t$ denote the translation along trajectory operators associated with (21) and (22), respectively, and $\mathrm{Ind}_C(\cdot, \cdot)$ is the topological index for k-set contractions. From this, we derive the *continuation principle*:

$$\mathrm{Ind}_C(\Phi_T, Z) = \mathrm{Ind}_C(\Psi_T^{(1)}, Z) = \lim_{t \to 0^+} \mathrm{Ind}_C(\widehat{\Psi}_t, Z). \tag{24}$$

We further establish *a priori* estimates for εT-periodic mild solutions of (21) assuming either

(i) the non-resonance condition at infinity (see (15)) and the absence of nonzero εT-periodic mild solutions of the linearized-at-infinity equation (16), or

(ii) the non-resonance condition at zero (see (15)) and the absence of nonzero εT-periodic mild solutions of the linearized-at-zero equation (17).

These estimates will enable us to apply the continuation principle (24). In order to compute the topological fixed-point index of the translation operator $\widehat{\Psi}_t$, for sufficiently small $t > 0$, we derive the linearization formulae for the autonomous nonlinear damped wave equation. The chapter concludes with the proof of Theorem II and its application to equation (1) with the Coulomb potential on $\mathbb{R}^3$ and suitable nonlinearities.

The Appendix gathers essential background material, including facts from the theory of linear operators on Banach spaces, Lebesgue and Sobolev spaces, measure of non-compactness, and the topological degree and fixed-point index for various classes of maps.

The main contributions of this thesis include:

- establishing general continuity result concerning the convergence of mild solutions to Cauchy problems (Theorem 1.3.1);
- defining an equivalent norm $\|\cdot\|_{s,V}$ on the space $\mathbb{X}$ (Theorem 2.3.3) and proving that the C_0-semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$ generated by the damped wave operator $\mathbb{A}$ is a k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$ for a properly chosen $s > 0$ (Theorem 2.3.9);
- deriving linear topological fixed-point index formulae (Theorem 2.4.4 and Corollary 2.4.5);
- providing an estimate of the measure of non-compactness in $L^2(\mathbb{R}^N)$ for sequences of functions with sublinear growth (Proposition 3.3.1);
- establishing a topological fixed-point index formula for the translation along trajectories operator associated with the autonomous evolution equation at resonance at infinity (Theorem 4.1.1) along with the resonant averaging principle (Theorem 4.2.1);
- providing a criterion for the existence of T-periodic mild solutions to the nonlinear damped wave equations at resonance at infinity (Theorem I);
- deriving an index averaging formula for the translation along trajectories operator associated with the damped wave equation under non-resonance conditions (Theorem 5.2.1);
- establishing topological fixed-point index formulae for autonomous damped wave equations in the non-resonance case (Theorem 5.4.1);
- providing a criterion for the existence of T-periodic mild solutions to the nonlinear damped wave equations under non-resonance conditions (Theorem II).

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Toruń, September 2025 *Władysław Klinikowski*

Notation

- We assume that all spaces considered are over the real numbers. When dealing with the complex spectrum or resolvent set, we consider the complexified space (see Appendix A.1).
- $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$, $D(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| \leq r\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$.
- $\mathbf{1}_X$ denotes the characteristic function of a subset $X \subset Y$.
- $\mathrm{Id} : X \to X$ denotes the identity map.
- $|U|$ denotes the Lebesgue measure of the set $U \subset \mathbb{R}^N$.
- $\|\cdot\|_X$ denotes the norm, and $\langle \cdot, \cdot \rangle_X$ the scalar product in the space X.
- $B_X(x_0, r) = \{x \in X : \|x - x_0\| < r\}$, where $(X, \|\cdot\|)$ is a normed space.
- $\mathrm{diam}\, U = \sup\{\|x - y\| : x, y \in U\}$ is the diameter of a subset U of normed space X.
- $\overline{U}$ denotes the closure. To emphasize that the closure is taken in X, we write $\overline{U}^X$.
- $\mathrm{conv}(U)$ denotes the convex hull and $\overline{\mathrm{conv}}(U)$ denotes the closed convex hull.
- $u \perp v$ means that vectors u and v are perpendicular in the unitary space $(X, \langle \cdot, \cdot \rangle)$; $U \perp V$ means that subspaces U, $V \subset X$ are orthogonal; $W^\perp$ denotes the orthogonal complement of the subset $W \subset X$.
- $\mathrm{Fix}(F, U) = \{x \in U : F(x) = x\}$, where $F : W \to X$ and $U \subset W \subset X$; $\mathrm{Fix}(F) = \mathrm{Fix}(F, X)$.
- $\deg_B(F, U)$ denotes the Brouwer topological degree of the map F on the set U (and with respect to the point 0) – see Theorem A.4.1.
- $\mathrm{Ind}_B(\cdot, \cdot)$ denotes the Brouwer topological index – see Theorem A.4.2.
- $\mathrm{Ind}_{LS}(\cdot, \cdot)$ denotes the Leray-Schauder topological index – see Theorem A.4.3.
- χ denotes the Hausdorff measure of non-compactness.
- $\mathrm{Ind}_C(\cdot, \cdot)$ denotes the topological index for k-set contractions – see Definition A.4.7.
- $m_{\mathrm{alg}}(\lambda)$ denotes algebraic multiplicity of the eigenvalue λ.
- $f|_{X_0}$ denotes the restriction of a map $f : X \to Y$ to a subset $X_0 \subset X$.
- If $A : D(A) \subset X \to X$ is a linear operator on a normed space X and $Y \subset X$ is invariant under A, then the part of A in Y is called the restriction of A to Y and is denoted by $A|_Y$ (see the discussion on p. 3).
- $\mathrm{Gr}A$ denotes the graph of the linear operator $A : D(A) \subset X \to X$ and $\|\cdot\|_A$ denotes the graph norm on $D(A)$ – see Definition A.1.3 (i).
- For a linear operator $A : D(A) \subset X \to X$, $\sigma_{\mathrm{ess}}(A)$ denotes the essential spectrum, $\sigma_{\mathrm{disc}}(A)$ the discrete spectrum, and $\sigma_p(A)$ the point spectrum. The resolvent set of A is denoted by $\rho(A)$, and $R(\lambda, A) = (\lambda I - A)^{-1} : X \to X$ – cf. Appendix A.1.
- $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators $A : X \to Y$, where X and Y are normed spaces, and $\mathcal{L}(X) = \mathcal{L}(X, X)$. If, on $(X, \|\cdot\|)$, we also consider another norm $\|\cdot\|_1$, then $\mathcal{L}(X, \|\cdot\|_1)$ denotes the space of bounded linear operators $A : (X, \|\cdot\|_1) \to (X, \|\cdot\|_1)$.

- $U_1 + U_2$ denotes the algebraic sum of subsets U_1, $U_2 \subset X$, and $X_1 \oplus X_2$ denotes the direct sum of linear subspaces $X_1 \subset X$ and $X_2 \subset X$. If the subspaces X_1 and X_2 are additionally closed, then $X_1 \oplus X_2$ is said to be the topological direct sum.

- $C(X,Y)$ denotes the space of continuous maps $f : X \to Y$, where X, Y are normed spaces, equipped with the supremum norm.

- $C^\infty(U)$ denotes the space of smooth functions on $U \subset \mathbb{R}^N$, while $C_0^\infty(U)$ denotes the space of smooth functions on U with compact support.

- $L^p(U)$ denotes the Lebesgue space. If $U = \mathbb{R}^N$, then $\|u\|_{L^p} = \|u\|_{L^p(\mathbb{R}^N)}$, $\langle u, v\rangle_{L^2} = \langle u, v\rangle_{L^2(\mathbb{R}^N)}$ and $B_{L^p}(x_0, r) = B_{L^p(\mathbb{R}^N)}(x_0, r)$. $L^p_{loc}(U)$ denotes the local Lebesgue space.

- $W^{k,p}(U)$ denotes the Sobolev space, where k is the maximal order of the weak derivative and p is the exponent of $L^p(U)$. We write $H^k(U) = W^{k,2}(U)$. If $U = \mathbb{R}^N$, then $\|u\|_{W^{k,p}} = \|u\|_{W^{k,p}(\mathbb{R}^N)}$, $\langle u, v\rangle_{H^k} = \langle u, v\rangle_{H^k(\mathbb{R}^N)}$, and $B_{W^{k,p}}(x_0, r) = B_{W^{k,p}(\mathbb{R}^N)}(x_0, r)$. The spaces $W^{k,p}_{loc}(U)$ and $H^k_{loc}(U)$ denote the corresponding local Sobolev spaces.

- $\{f > 0\} = \{x \in \mathbb{R}^N : f(x) > 0\}$, where $f : \mathbb{R}^N \to \mathbb{R}$ is a function. The sets $\{f = 0\}$ and $\{f < 0\}$ are defined analogously.

- $[a]$ denotes the floor function at $a \in \mathbb{R}$, i.e., the greatest integer less than or equal to a.

Chapter 1

General continuity and averaging principle

We begin by recalling the basic properties of C_0-semigroups and their generators, along with classical results from this theory. We then discuss the existence and uniqueness of solutions to semilinear evolution equations, as well as their continuous dependence on the initial data and the parametrized nonlinear term.

Next, inspired by a result of D. Henry [35, Thm. 3.4.8], we prove the main result of this chapter: a general continuity theorem for a sequence of evolution equations under the assumption of integral convergence of the nonlinear terms. From this, we first derive continuity results for semilinear equations with frequency changing nonlinearities, and then establish the abstract averaging principle. We apply these results in Chapter 5.

In the concluding section, we examine the existence, continuity, and compactness properties of the translation along trajectories operator.

1.1 C_0-semigroups of bounded linear operators

Assume that $(X, \|\cdot\|)$ is a Banach space. A family of bounded linear operators $\{T(t) : X \to X\}_{t\geq 0}$ is called a C_0-*semigroup* (or *strongly continuous semigroup*) if

(i) (semigroup property) $T(0) = I$, $T(t+s) = T(t)T(s)$ for t, $s \geq 0$;

(ii) (strong continuity property) $\lim_{t\to 0^+} T(t)u = u$ for $u \in X$.

A linear operator $A : D(A) \subset X \to X$ is said to be the (*infinitesimal*) *generator* of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ if

$$D(A) = \left\{ u \in X : \lim_{h\to 0^+} \frac{T(h)u - u}{h} \text{ exists} \right\}$$

and

$$Au = \lim_{h\to 0^+} \frac{T(h)u - u}{h} \quad \text{for } u \in D(A).$$

If A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then there are no other C_0-semigroups generated by A, i.e. A determines $\{T(t)\}_{t\geq 0}$ uniquely – see [47, Ch. 1, Thm. 2.6]. We denote this semigroup by $\{e^{tA}\}_{t\geq 0}$.

Remark 1.1.1. It is well-known that for any C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ generated by A there exist numbers $K \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq K e^{\omega t} \quad \text{for } t \geq 0 \tag{1.1}$$

– see [47, Ch. 1., Thm. 2.2]. If $K = 1$ and $\omega \leq 0$, then $\{e^{tA}\}_{t\geq 0}$ is called a C_0-*semigroup of contractions*. Moreover, for any $u \in X$, the map $t \mapsto e^{tA}u$ is a continuous function from $[0, +\infty)$ into X, cf. [47, Ch. 1, Corollary 2.3]. □

In the following theorem we recall basic facts about generators of C_0-semigroups.

Theorem 1.1.2. [47, Ch. 1, Corollary 2.5, Thm. 2.4] *If $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$, then*

(i) *A is closed and $D(A)$ is dense in X;*

(ii) *for any $u \in D(A)$ and $t \geq 0$, $e^{tA}u \in D(A)$ and*

$$\frac{d}{dt}(e^{tA}u) = Ae^{tA}u = e^{tA}Au;$$

(iii) *for any $u \in X$ and $t \geq 0$, $\int_0^t e^{sA}u\,ds \in D(A)$ and*

$$A\int_0^t e^{sA}u\,ds = e^{tA}u - u.$$

Let us provide classical results about characterization of generators of C_0-semigroups.

Theorem 1.1.3. [Lumer-Phillips] [12, Thm. 3.4.4] *A linear operator $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of contractions if and only if $D(A)$ is dense in X and A is a maximal dissipative operator.*

Remark 1.1.4.

(i) Operator A is called *dissipative* if $\|u - \lambda Au\| \geq \|u\|$, for all $\lambda > 0$ and $u \in D(A)$, and it is called *maximal dissipative* if, in addition, $\mathrm{Im}(I - \lambda A) = X$ for all $\lambda > 0$.

(ii) Let $A : D(A) \subset X \to X$ be a dissipative operator. If there exists $\lambda_0 > 0$ such that $\mathrm{Im}(I - \lambda_0 A) = X$, then $\mathrm{Im}(I - \lambda A) = X$ for any $\lambda > 0$ (see [12, Prop. 2.2.6]), i.e., A is a maximal dissipative operator.

(iii) It is easily seen that a linear operator $A : D(A) \subset X \to X$ is maximal dissipative if and only if $(0, +\infty) \subset \rho(A)$ and

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq 1/\lambda \quad \text{for } \lambda > 0. \tag{1.2}$$

Thus, in view of the Lumer-Phillips Theorem, a linear operator A is the generator of a C_0-semigroup of contractions if and only if $D(A)$ is dense in X, $(0, +\infty) \subset \rho(A)$ and $\|R(\lambda, A)\| \leq 1/\lambda$ for all $\lambda > 0$ (see [47, Ch. 1, Thm. 3.1]).

(iv) If X is a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, one can show that a linear operator $A : D(A) \subset X \to X$ is dissipative if and only if $\langle Au, u \rangle \leq 0$ for $u \in D(A)$, cf. [12, Prop. 2.4.2]. Moreover, if A is a maximal dissipative operator, then $D(A)$ is dense in X, compare [12, Corollary 2.4.3]. Hence, by the Lumer-Phillips Theorem, A is the generator of a C_0-semigroup of contractions on a real Hilbert space X if and only if $\langle Au, u \rangle \leq 0$ for $u \in D(A)$ and $\mathrm{Im}(\lambda I - A) = X$ for some $\lambda > 0$. □

Theorem 1.1.5. [Hille-Yosida] [47, Ch. 1, Thm. 5.3] *A linear operator $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ satisfying* (1.1) *for all $t \geq 0$ and some $K \geq 1$, $\omega \in \mathbb{R}$, if and only if $D(A)$ is dense in X, $(\omega, +\infty) \subset \rho(A)$ and*

$$\|R(\lambda, A)^n\|_{\mathcal{L}(X)} \leq \frac{K}{(\lambda - \omega)^n} \quad \text{for } \lambda > \omega \text{ and } n \geq 1$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.

Remark 1.1.6. The result in Remark 1.1.4 (iii) is often called the Hille-Yosida Theorem, while the general case from Theorem 1.1.5 is stated as the Feller-Miyadera-Phillips Theorem. □

We present now two perturbation results. The first says that the generator of a C_0-semigroup perturbed by a relatively bounded linear operator is still the generator of some C_0-semigroup. The second one provides the integral formula for a C_0-semigroup generated by the operator perturbed by a bounded linear operator.

Theorem 1.1.7. [25, Ch. III, Sec. 1, Corollary 1.5] *Assume that $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup, and that the linear operator $B : D(B) \subset X \to X$ is relatively A-bounded, that is, $D(A) \subset D(B)$ and the linear operator $B : (D(A), \|\cdot\|_A) \to (X, \|\cdot\|)$ is bounded, where $\|u\|_A = (\|u\|^2 + \|Au\|^2)^{1/2}$ is the graph norm on $D(A)$. Then the linear operator $A+B : D(A+B) \subset X \to X$, given by*

$$(A+B)u = Au + Bu \quad \textit{for } u \in D(A+B) = D(A), \tag{1.3}$$

is the generator of a C_0-semigroup.

Theorem 1.1.8. [see [25, Ch. III, Sec. 1, Thm. 1.3 and Corollary 1.7]] *Suppose that $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ and $B : X \to X$ is a bounded linear operator. Then, $A+B : D(A+B) \subset X \to X$, defined as in* (1.3)*, is the generator of a C_0-semigroup $\{e^{t(A+B)}\}_{t\geq 0}$ and, for all $t \geq 0$ and $u \in X$,*

$$e^{t(A+B)}u = e^{tA}u + \int_0^t e^{(t-\tau)A}Be^{\tau(A+B)}u \, d\tau.$$

Next, we introduce a result showing that if a subspace is invariant under the generator of a semigroup, then it is also invariant under the semigroup itself. To this end, let us recall that for a subspace (not necessarily closed) Y of X and a linear operator $A : D(A) \subset X \to X$, we define a new linear operator $\widetilde{A} : D(\widetilde{A}) \subset Y \to Y$ (where, in particular, $D(A)$ may be the whole space X), called the *part of A in Y*, as follows:

$$\widetilde{A}u = Au \quad \text{for} \quad u \in D(\widetilde{A}) = \{u \in D(A) \cap Y : Au \in Y\}.$$

If the subspace Y of X is additionally invariant under A, that is, $A(D(A) \cap Y) \subset Y$, then we call the operator $\widetilde{A}$ the *restriction of A to Y* and denote it by $A|_Y$.

Proposition 1.1.9. [compare with [25, Ch. II, Subsection 2.3]] *Let $A : D(A) \subset X \to X$ be the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space X. We assume that X decomposes as a topological direct sum $X = X_P \oplus X_Q$, with the corresponding projections P, Q, and that the subspaces X_P, X_Q are invariant under A.*

Then, for all $t \geq 0$, X_P and X_Q are invariant under e^{tA}, and e^{tA} commutes with the projections P and Q. Furthermore, the restricted operators $A|_{X_P}$, $A|_{X_Q}$ are the generators of C_0-semigroups on the subspaces X_P, X_Q, respectively, and

$$e^{tA|_{X_P}} = e^{tA}\big|_{X_P}, \quad e^{tA|_{X_Q}} = e^{tA}\big|_{X_Q} \quad \textit{for all } t \geq 0.$$

1.2 Existence and uniqueness of mild solutions

Let $(X, \|\cdot\|)$ be a Banach space, $A : D(A) \subset X \to X$ be the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ and $f \in L^1_{loc}([0,\tau), X)$, where $0 < \tau \leq +\infty$ and $L^1_{loc}([0,\tau), X)$ is a space consisting of locally Bochner integrable functions $f : [0,\tau) \to X$ (cf. [12, Sec. 1.4.2]). We consider the following Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), & t \in (0,\tau), \\ u(0) = \overline{u} \in X. \end{cases} \tag{1.4}$$

We say that a function $u : [0,\tau) \to X$ is a *mild solution* of (1.4) if it satisfies the integral equation, known as the *Duhamel formula*:

$$u(t) = e^{tA}\,\overline{u} + \int_0^t e^{(t-s)A} f(s)\, ds \quad \text{for } t \in [0,\tau) \tag{1.5}$$

where the integral above is understood in the sense of the Bochner integral. If $\tau = +\infty$, the mild solution $u : [0,+\infty) \to X$ of the problem (1.4) is called global. By [12, Lemma 4.1.5], the mild solution u is continuous.

We will make use of the following continuity and compactness result related to (1.4).

Proposition 1.2.1. *Assume that $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$, that $f_n \in L^1_{loc}([0,\tau), X)$ for every $n \geq 1$, where $0 < \tau \leq +\infty$, that $(\overline{u}_n)_{n\geq 1}$ is a sequence in X, and that $u_n : [0,\tau) \to X$, $n \geq 1$, are mild solutions of the problem (1.4) with $f = f_n$ and $\overline{u} = \overline{u}_n$.*

(i) [14, Proposition 2.5] *If $\overline{u}_n \to \overline{u}_0$ in X and, for any $0 < T < \tau$, $f_n \to f_0$ in $L^1([0,T], X)$, then $u_n(t) \xrightarrow{n\to\infty} u_0(t)$ uniformly on bounded subsets of $[0,\tau)$ with respect to t, where $u_0 : [0,\tau) \to X$ is the mild solution of (1.4) with $f = f_0$ and $\overline{u} = \overline{u}_0$.*

(ii) [14, Proposition 2.7] *If a bounded interval $[0,T] \subset [0,\tau)$ is such that functions f_n, $n \geq 1$, are uniformly bounded on $[0,T]$ and, for any $t \in [0,T]$, the set $\{u_n(t)\}_{n\geq 1}$ is relatively compact in X, then the set $\left\{ u_n|_{[0,T]} \right\}_{n\geq 1}$ is relatively compact in $C([0,T], X)$.*

Now suppose that $F : [0,+\infty) \times X \to X$ is continuous and

(F1) F satisfies the (uniform) Lipschitz condition, that is, there exists a constant $L > 0$ such that, for all $t \geq 0$ and u_1, $u_2 \in X$,

$$\|F(t,u_1) - F(t,u_2)\| \leq L\|u_1 - u_2\|;$$

(F2) F is (uniformly) bounded at zero, that is, there exists a constant $M_0 > 0$ such that $\|F(t,0)\| \leq M_0$ for all $t \geq 0$.

Observe that conditions (F1) and (F2) imply that F has a sublinear growth, that is, for all $t \geq 0$ and $u \in X$,

$$\|F(t,u)\| \leq L\|u\| + \|F(t,0)\| \leq C(\|u\| + 1) \tag{1.6}$$

for $C = \max\{L, M_0\}$.

Consider the problem

$$P(A,F,\overline{u}) \qquad \begin{cases} \dot{u}(t) &= Au(t) + F(t,u(t)), \quad t \in (0,\tau), \\ u(0) &= \overline{u} \in X \end{cases}$$

where $0 < \tau \leq +\infty$. We say that a continuous function $u : [0,\tau) \to X$ is a mild solution of $P(A,F,\overline{u})$ if u satisfies the Duhamel formula (1.5) with $f : [0,\tau) \to X$ defined by $f(t) = F(t,u(t))$, for any $t \in [0,\tau)$. It is well-known (see [47, Ch. 6, Thm. 1.4]) that the local existence and uniqueness result holds, i.e. if F is continuous and locally Lipschitz([1]), then for any $\overline{u} \in X$ there exists a unique mild solution $u : [0,\tau_{\max}) \to X$ of $P(A,F,\overline{u})$ on a maximal interval $[0,\tau_{\max})$. In addition, if $\tau_{\max} < +\infty$, one has

$$\lim_{t\to\tau_{\max}^+} \|u(t)\| = +\infty. \tag{1.7}$$

([1])A mapping $F : [0,+\infty) \times X \to X$ is called locally Lipschitz if for any $T_0 > 0$ and $R_0 > 0$ there exists $L = L(T_0,R_0) > 0$ such that

$$\|F(t,u_1) - F(t,u_2)\| \leq L\|u_1 - u_2\|$$

for $t \in [0,T_0]$ and u_1, $u_2 \in B_X(0,R_0)$.

We have also the continuous dependence on initial conditions, that is, if $\overline{u}_n \to \overline{u}_0$ in X and $u_n : [0, \tau) \to X$, $n \geq 0$, are mild solutions of $P(A, F, \overline{u}_n)$, then $u_n(t) \to u_0(t)$ uniformly for t on bounded subsets of $[0, \tau)$, see [47, Ch. 6, Thm. 1.2].

Let a continuous map $F : [0, +\infty) \times X \to X$ satisfy conditions (F1) and (F2), and $u(\cdot, \overline{u}) : [0, \tau_{\max}) \to X$ be the mild solution of $P(A, F, \overline{u})$. We claim that $\tau_{\max} = +\infty$. Indeed, suppose to the contrary that $\tau_{\max} < +\infty$. Applying the Duhamel formula (1.5) and inequalities (1.1) and (1.6) we get, for $t \in [0, \tau_{\max})$,

$$\begin{aligned} \|u(t, \overline{u})\| &\leq K e^{|\omega|\tau_{\max}} \|\overline{u}\| + \int_0^t K e^{|\omega|\tau_{\max}} \|F(s, u(s, \overline{u}))\| \, ds \\ &\leq K e^{|\omega|\tau_{\max}} \|\overline{u}\| + \int_0^t K e^{|\omega|\tau_{\max}} C(1 + \|u(s, \overline{u})\|) \, ds \\ &\leq C_1 + C_2 \int_0^t \|u(s, \overline{u})\| \, ds \end{aligned} \tag{1.8}$$

where $C_1 = K e^{|\omega|\tau_{\max}} (\|\overline{u}\| + \tau_{\max} C)$ and $C_2 = K e^{|\omega|\tau_{\max}} C$. Hence, based on the Gronwall inequality (Lemma A.2.1 (ii)), we obtain, for any $t \in [0, \tau_{\max})$,

$$\|u(t, \overline{u})\| \leq C_1 e^{C_2 t} \leq C_1 e^{C_2 \tau_{\max}},$$

a contradiction with (1.7).

Remark 1.2.2. Assume that a continuous map $F : [0, +\infty) \times X \to X$ satisfies (F1) and (F2), and $u(\cdot, \overline{u}) : [0, \tau) \to X$ is the mild solution of $P(A, F, \overline{u})$. Then the set $\{u(t, \overline{u}) : t \in [0, T_0], \overline{u} \in B_X(0, R_0)\}$ is bounded in X, for any $0 < T_0 < \tau$ and $R_0 > 0$. Indeed, using (1.8), we get, for all $t \in [0, T_0]$ and $\overline{u} \in B_X(0, R_0)$,

$$\|u(t, \overline{u})\| \leq C_1 + C_2 \int_0^t \|u(s, \overline{u})\| \, ds$$

where $C_1 = K e^{|\omega| T_0} (R_0 + T_0 C)$ and $C_2 = K e^{|\omega| T_0} C$. Consequently, based on the Gronwall inequality, we obtain

$$\|u(t, \overline{u})\| \leq C_1 e^{C_2 T_0}$$

for all $t \in [0, T_0]$ and $\overline{u} \in B_X(0, R_0)$. □

The following fact concerns constant mild solutions for the autonomous problem.

Lemma 1.2.3. [cf. [14, Lemma 3.1]] *Let $A : D(A) \subset X \to X$ be the generator of a C_0-semigroup $\{e^{tA}\}_{t \geq 0}$ and $F : X \to X$ be continuous. Suppose that $u : [0, +\infty) \to X$, $u(t) = \overline{u} \in X$, for all $t \geq 0$, is a mild solution of*

$$\begin{cases} \dot{u}(t) = Au(t) + F(u(t)), & t > 0, \\ u(0) = \overline{u}. \end{cases} \tag{1.9}$$

Then $\overline{u} \in D(A)$ and

$$A\overline{u} + F(\overline{u}) = 0. \tag{1.10}$$

Conversely, if $\overline{u} \in D(A)$ satisfies (1.10), *then the constant function $u(t) = \overline{u}$, for all $t \geq 0$, is a mild solution of* (1.9).

1.3 Continuity for nonlinear evolution equations

We are now ready to formulate the main result of this chapter – namely, the convergence of mild solutions of a sequence of Cauchy problems under the assumption of integral convergence of the nonlinear terms. Our theorem extends the first result of this kind, due to D. Henry [35, Thm. 3.4.8], and we will follow his approach in the proof. The assumption on the nonlinear terms is essentially weaker than pointwise convergence or convergence in the space of locally Bochner integrable functions, and it is needed to establish the abstract averaging principle, which will be applied in the non-resonant case.

Theorem 1.3.1. *Assume that X is a separable Banach space and mappings $F_n : [0,+\infty)\times X \to X$, $n \geq 0$, are continuous and satisfy conditions* (F1) *and* (F2) *(see p. 4) with constants independent of n. Moreover, suppose that $\overline{u}_n \to \overline{u}_0$ in X, $\{F_n(t,u)\}_{n\geq 1}$ is relatively compact, for all $t \geq 0$ and $u \in X$, and*

$$\int_0^t F_n(\tau,u)\,d\tau \to \int_0^t F_0(\tau,u)\,d\tau \quad \text{as } n\to\infty \tag{1.11}$$

for any $t > 0$ and $u \in X$.

If $u_n : [0,+\infty) \to X$, $n \geq 0$, are the mild solutions of $P(A, F_n, \overline{u}_n)$ (see p. 4), then $u_n(t) \to u_0(t)$ uniformly for t on bounded subsets of $[0,+\infty)$*.*

Before the proof we need some preparatory facts. Our first result provides a sufficient condition for convergence of a sequence in X.

Lemma 1.3.2. *If a sequence $(y_n)_{n\geq 1}$ in X is such that the set $\{y_n\}_{n\geq 1}$ is relatively compact in X, and there exist $\lambda \in \rho(A)$ and $y_0 \in X$ such that*

$$R(\lambda,A)y_n \to R(\lambda,A)y_0 \quad \text{as } n\to\infty, \tag{1.12}$$

then

$$y_n \to y_0.$$

Proof. Suppose, to the contrary, that $(y_n)_{n\geq 1}$ does not converge to y_0. Hence, there exist $\varepsilon > 0$ and subsequence $(y_{n_k})_{k\geq 1}$ such that

$$\|y_{n_k} - y_0\| \geq \varepsilon \ \text{ for all } k \geq 1. \tag{1.13}$$

By assumption, there exists a subsequence $(y_{n_{k_l}})_{l\geq 1}$ such that $y_{n_{k_l}} \to y_0'$ for some $y_0' \in X$. Since the resolvent of an operator is a bounded linear operator, we get

$$R(\lambda,A)y_{n_{k_l}} \to R(\lambda,A)y_0' \ \text{ as } l\to\infty$$

where $\lambda \in \rho(A)$ is as in (1.12). By the assumed property (1.12) we have $R(\lambda,A)y_0' = R(\lambda,A)y_0$ and consequently, because $R(\lambda,A)$ is injective, $y_0' = y_0$. Hence, $y_{n_{k_l}} \to y_0$ and this contradicts (1.13). The proof is completed. □

The next lemma concerns differentation of semigroups.

Lemma 1.3.3. *Assume that $\{S(t) : X \to X\}_{t\geq 0}$, is a family of bounded linear operators such that, for any bounded subset $D \subset [0,+\infty)$,*

$$\sup_{t\in D} \|S(t)\|_{\mathcal{L}(X)} < +\infty$$

and, for any $u \in X$, the function

$$[0,+\infty) \ni t \mapsto S(t)u \in X \tag{1.14}$$

is continuous. Suppose that there exists a non-empty subspace $X_0 \subset X$ such that, for any $u \in X_0$, the map given by (1.14) *is differentiable (at* $t = 0$ *we consider the right-hand derivative) and let $f : [0, +\infty) \to X_0 \subset X$ be a differentiable function. Then, for any $t \geq 0$,*

$$\frac{d}{dt}(S(t)f(t)) = \frac{d}{dt}(S(t)v)\Big|_{v=f(t)} + S(t)\left(\frac{d}{dt}f(t)\right).$$

Proof. Set $t \geq 0$ and then, by assumption, $\frac{d}{dt}f(t)$ and $\frac{d}{dt}\left(S(t)v\right)\big|_{v=f(t)}$ exist. Take $h \neq 0$ such that $|h| \leq 1$ and $t + h \geq 0$. We have

$$\begin{aligned}\frac{S(t+h)f(t+h) - S(t)f(t)}{h} &= S(t+h)\left(\frac{f(t+h) - f(t)}{h} - \frac{d}{dt}f(t)\right) \\ &\quad + S(t+h)\left(\frac{d}{dt}f(t)\right) + \frac{S(t+h) - S(t)}{h}f(t).\end{aligned}$$

Let us put

$$C = \sup\{\|S(s)\|_{\mathcal{L}(X)} : |t - s| \leq 1\}$$

and, by assumption, $C < +\infty$. We see that

$$\left\|S(t+h)\left(\frac{f(t+h) - f(t)}{h} - \frac{d}{dt}f(t)\right)\right\| \leq C\left\|\frac{f(t+h) - f(t)}{h} - \frac{d}{dt}f(t)\right\| \to 0 \quad \text{as } h \to 0,$$

$$\frac{S(t+h) - S(t)}{h}f(t) \to \frac{d}{dt}(S(t)v)\Big|_{v=f(t)} \qquad \text{as } h \to 0$$

and

$$S(t+h)\left(\frac{d}{dt}f(t)\right) \to S(t)\left(\frac{d}{dt}f(t)\right) \quad \text{as } h \to 0.$$

This proves the assertion. □

Now, we provide an integral formula.

Lemma 1.3.4. *If $f : [0, +\infty) \to X$ is a continuous function such that, for all $t \geq 0$,*

$$\int_0^t f(\tau)\,d\tau \in D(A)$$

and the function

$$[0, +\infty) \ni t \mapsto A\int_0^t f(\tau)\,d\tau \in X \tag{1.15}$$

is continuous, then we have, for any $t \geq 0$,

$$\int_0^t e^{(t-\tau)A}f(\tau)\,d\tau = \int_0^t f(\tau)\,d\tau + \int_0^t e^{(t-\tau)A}A\int_0^\tau f(\sigma)\,d\sigma\,d\tau. \tag{1.16}$$

Proof. Since the thesis is obviously satisfied for $t = 0$, we shall assume that $t > 0$. Let $g : [0, +\infty) \to X$, $g(\tau) = \int_0^\tau f(\sigma)\,d\sigma$. By the continuity of f, g is differentiable and, by Theorem 1.1.2 (ii), the map

$$[0, +\infty) \ni \tau \mapsto e^{\tau A}u \in X$$

is differentiable for any $u \in D(A)$. Moreover, due to assumption, $g([0,+\infty)) \subset D(A)$. Therefore, we may apply Lemma 1.3.3, which together with Theorem 1.1.2 (ii) implies

$$\begin{aligned} e^{(t-\tau)A} f(\tau) &= e^{(t-\tau)A}\left(\frac{d}{d\tau} g(\tau)\right) \\ &= \frac{d}{d\tau}\left(e^{(t-\tau)A} g(\tau)\right) - \frac{d}{d\tau}\left(e^{(t-\tau)A} v\right)\Big|_{v=g(\tau)} \\ &= \frac{d}{d\tau}\left(e^{(t-\tau)A}\int_0^\tau f(\sigma)\,d\sigma\right) + Ae^{(t-\tau)A}\int_0^\tau f(\sigma)\,d\sigma \\ &= \frac{d}{d\tau}\left(e^{(t-\tau)A}\int_0^\tau f(\sigma)\,d\sigma\right) + e^{(t-\tau)A}A\int_0^\tau f(\sigma)\,d\sigma, \end{aligned} \tag{1.17}$$

which holds for all $\tau \in [0,t]$. Furthermore, since function given by (1.15) is assumed to be continuous, the map

$$[0,t] \ni \tau \mapsto e^{(t-\tau)A} A \int_0^\tau f(\sigma)\,d\sigma \in X$$

is also continuous, thus, integrating (1.17) from $\tau = 0$ to $\tau = t$, we arrive at (1.16). □

The lemma below is related to the condition (1.11).

Lemma 1.3.5. *If mappings $F_n : [0,+\infty) \times X \to X$, $n \geq 0$, are continuous and satisfy* (F1) *(see p. 4) with a constant $L > 0$ independent of n, and F_n, $n \geq 0$, satisfy the property* (1.11), *for all $t > 0$ and $\overline{u} \in X$, then, for any continuous function $u : [0,+\infty) \to X$ and $t > 0$,*

$$\int_0^t F_n(\tau, u(\tau))\,d\tau \to \int_0^t F_0(\tau, u(\tau))\,d\tau \quad \text{as } n \to \infty.$$

Proof. Let $u : [0,+\infty) \to X$ be continuous and fix some $t > 0$ and $\varepsilon > 0$. Then $u : [0,t] \to X$ is uniformly continuous, hence there exists $\delta > 0$ such that, for s_1, $s_2 \in [0,t]$ fulfilling $|s_1 - s_2| < \delta$,

$$\|u(s_1) - u(s_2)\| < \varepsilon/3Lt. \tag{1.18}$$

Let $0 = s_0 < s_1 < \ldots < s_N = t$ and $0 < s_{j+1} - s_j < \delta$ for $j = 0,\ldots,N-1$. By assumption there exists n_0 such that, for all $n \geq n_0$ and $j = 0,\ldots,N-1$,

$$\left\|\int_0^{s_j} F_n(\tau, u(s_j))\,d\tau - \int_0^{s_j} F_0(\tau, u(s_j))\,d\tau\right\| < \varepsilon/6N$$

and

$$\left\|\int_0^{s_{j+1}} F_n(\tau, u(s_j))\,d\tau - \int_0^{s_{j+1}} F_0(\tau, u(s_j))\,d\tau\right\| < \varepsilon/6N$$

that is

$$\left\|\int_{s_j}^{s_{j+1}} F_n(\tau, u(s_j))\,d\tau - \int_{s_j}^{s_{j+1}} F_0(\tau, u(s_j))\,d\tau\right\| < \varepsilon/3N. \tag{1.19}$$

Next, we define $\widetilde{u}_\varepsilon : [0,t] \to X$,

$$\widetilde{u}_\varepsilon(\tau) = \sum_{j=0}^{N-1} u(s_j)\mathbf{1}_{[s_j,s_{j+1})}(\tau).$$

By (F1) and (1.18), one has, for any $n \geq 0$,

$$\begin{aligned} \left\|\int_0^t F_n(\tau, u(\tau))\,d\tau - \int_0^t F_n(\tau, \widetilde{u}_\varepsilon(\tau))\,d\tau\right\| &\leq \int_0^t \|F_n(\tau, u(\tau)) - F_n(\tau, \widetilde{u}_\varepsilon(\tau))\|\,d\tau \\ &= \sum_{j=0}^{N-1}\int_{s_j}^{s_{j+1}} \|F_n(\tau, u(\tau)) - F_n(\tau, u(s_j))\|\,d\tau \\ &\leq \sum_{j=0}^{N-1}\int_{s_j}^{s_{j+1}} L\|u(\tau) - u(s_j)\|\,d\tau < \varepsilon/3. \end{aligned} \tag{1.20}$$

Furthermore, in view of (1.19), we see that

$$\begin{aligned}&\left\|\int_0^t F_n(\tau,\widetilde{u}_\varepsilon(\tau))\,d\tau-\int_0^t F_0(\tau,\widetilde{u}_\varepsilon(\tau))\,d\tau\right\|\\&=\left\|\sum_{j=0}^{N-1}\left(\int_{s_j}^{s_{j+1}} F_n(\tau,u(s_j))\,d\tau-\int_{s_j}^{s_{j+1}} F_0(\tau,u(s_j))\,d\tau\right)\right\|<\frac{\varepsilon}{3N}\cdot N=\frac{\varepsilon}{3}.\end{aligned}$$

From this and (1.20) we deduce that, for $n\ge n_0$,

$$\begin{aligned}\left\|\int_0^t F_n(\tau,u(\tau))\,d\tau-\int_0^t F_0(\tau,u(\tau))\,d\tau\right\|&\le\left\|\int_0^t F_n(\tau,u(\tau))-F_n(\tau,\widetilde{u}_\varepsilon(\tau))\,d\tau\right\|\\&+\left\|\int_0^t F_n(\tau,\widetilde{u}_\varepsilon(\tau))-F_0(\tau,\widetilde{u}_\varepsilon(\tau))\,d\tau\right\|+\left\|\int_0^t F_0(\tau,\widetilde{u}_\varepsilon(\tau))-F_0(\tau,u(\tau))\,d\tau\right\|\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\end{aligned}$$

which ends the proof. □

Proof of Theorem 1.3.1. We take a bounded interval $[0,T_0]\subset[0,+\infty)$. In view of Remark 1.1.1, there exists $C_1>0$ such that

$$\|e^{tA}\|_{\mathcal{L}(X)}\le C_1\quad\text{for } t\in[0,T_0].$$

Since the operators $F_n:[0,+\infty)\times X\to X$, $n\ge 0$, fulfill (F1) and (F2) (see p. 4), with common constants, there exists a constant $C_2>0$ such that

$$\|F_n(t,u_0(t))\|\le C_2\quad\text{for all } n\ge 0 \text{ and } t\in[0,T_0].$$

The Duhamel formula (see (1.5)) yields, for any $n\ge 1$ and $t\in[0,T_0]$,

$$\begin{aligned}\|u_n(t)-u_0(t)\|&\le\|e^{tA}\,\overline{u}_n-e^{tA}\,\overline{u}_0\|\\&\quad+\left\|\int_0^t\left(e^{(t-\tau)A}\,F_n(\tau,u_n(\tau))-e^{(t-\tau)A}\,F_0(\tau,u_0(\tau))\right)d\tau\right\|\\&\le\|e^{tA}\,\overline{u}_n-e^{tA}\,\overline{u}_0\|\\&\quad+\left\|\int_0^t e^{(t-\tau)A}\left(F_n(\tau,u_n(\tau))-F_n(\tau,u_0(\tau))\right)d\tau\right\|\\&\quad+\left\|\int_0^t\left(e^{(t-\tau)A}\,F_n(\tau,u_0(\tau))-e^{(t-\tau)A}\,F_0(\tau,u_0(\tau))\right)d\tau\right\|.\end{aligned}\tag{1.21}$$

For any $n\ge 1$ we define $\gamma_n^{(1)}:[0,+\infty)\to\mathbb{R}$ and $\gamma_n^{(2)}:[0,+\infty)\to\mathbb{R}$ as follows

$$\gamma_n^{(1)}(t)=\|e^{tA}\,\overline{u}_n-e^{tA}\,\overline{u}_0\|,$$

$$\gamma_n^{(2)}(t)=\left\|\int_0^t\left(e^{(t-\tau)A}\,F_n(\tau,u_0(\tau))-e^{(t-\tau)A}\,F_0(\tau,u_0(\tau))\right)d\tau\right\|$$

and finally the function $\gamma_n:[0,+\infty)\to\mathbb{R}$

$$\gamma_n(t)=\gamma_n^{(1)}(t)+\gamma_n^{(2)}(t).$$

Due to (1.21) and the Lipschitz condition, one has, for any $n\ge 1$ and $t\in[0,T_0]$,

$$\begin{aligned}\|u_n(t)-u_0(t)\|&\le\gamma_n(t)+\left\|\int_0^t e^{(t-\tau)A}\left(F_n(\tau,u_n(\tau))-F_n(\tau,u_0(\tau))\right)d\tau\right\|\\&\le\gamma_n(t)+L\int_0^t\|e^{(t-\tau)A}\|_{\mathcal{L}(X)}\|u_n(\tau)-u_0(\tau)\|\,d\tau\\&=\gamma_n(t)+C\int_0^t\|u_n(\tau)-u_0(\tau)\|\,d\tau\end{aligned}\tag{1.22}$$

where $C = C_1 L$. Observe that, for all $n \geq 1$, the function γ_n is continuous. Therefore, based on the Gronwall inequality (see Lemma A.2.1 (ii)), we obtain, for any $n \geq 1$ and $t \in [0, T_0]$,

$$\begin{aligned} \|u_n(t) - u_0(t)\| &\leq \gamma_n(t) + C \int_0^t \gamma_n(\tau) e^{C(t-\tau)} \, d\tau \\ &\leq \gamma_n(t) + C e^{Ct} \int_0^t \gamma_n(\tau) \, d\tau \leq \gamma_n(t) + C e^{CT_0} \int_0^{T_0} \gamma_n(\tau) \, d\tau. \end{aligned} \tag{1.23}$$

We shall prove that, for any $t \in [0, T_0]$,

$$\gamma_n(t) \to 0 \quad \text{as } n \to \infty. \tag{1.24}$$

To this end, fix $t \in [0, T_0]$. Since $\{e^{tA}\}_{t \geq 0}$ is a C_0-semigroup, we readily see that

$$\gamma_n^{(1)}(t) \to 0 \quad \text{as } n \to \infty.$$

Therefore, it remains to show that

$$\gamma_n^{(2)}(t) \to 0 \quad \text{as } n \to \infty.$$

We put, for $n \geq 1$,

$$y_n(t) = \int_0^t e^{(t-\tau)A} \, F_n(\tau, u_0(\tau)) \, d\tau$$

and

$$y_0(t) = \int_0^t e^{(t-\tau)A} \, F_0(\tau, u_0(\tau)) \, d\tau.$$

Then

$$\gamma_n^{(2)}(t) = \|y_n(t) - y_0(t)\|. \tag{1.25}$$

We see that the maps

$$[0, t] \ni \tau \mapsto e^{(t-\tau)A} F_n(\tau, u_0(\tau)) \in X, \quad n \geq 1,$$

are continuous and

$$\left\| e^{(t-\tau)A} F_n(\tau, u_0(\tau)) \right\| \leq C_1 C_2 \quad \text{for all } n \geq 1 \text{ and } \tau \in [0, t].$$

By assumption, X is separable. Therefore, we can use the integral inequality for the Hausdorff measure of non-compactness χ (see Theorem A.3.3):

$$\chi(\{y_n(t)\}_{n \geq 1}) \leq \int_0^t \chi\left(\left\{ e^{(t-\tau)A} F_n(\tau, u_0(\tau)) \right\}_{n \geq 1} \right) d\tau. \tag{1.26}$$

Using the bounded linear operator estimate for the measure of non-compactness (cf. Proposition A.3.1 (v)), we have, for any $\tau \in [0, t]$,

$$\chi\left(\left\{ e^{(t-\tau)A} F_n(\tau, u_0(\tau)) \right\}_{n \geq 1} \right) \leq \|e^{(t-\tau)A}\|_{\mathcal{L}(X)} \, \chi\left(\{F_n(\tau, u_0(\tau))\}_{n \geq 1} \right) = 0, \tag{1.27}$$

because, by assumption, the set $\{F_n(\tau, u_0(\tau)) : n \geq 1\}$ is relatively compact for any $\tau \geq 0$. Taking into account (1.26) and (1.27) we conclude that

$$\chi\left(\{y_n(t)\}_{n \geq 1} \right) = 0 \tag{1.28}$$

i.e. $\{y_n(t)\}_{n \geq 1}$ is relatively compact in X.

Let $\lambda > \omega$. Since the linear operator $R(\lambda, A) : X \to X$ is bounded, one has, for $n \geq 0$,

$$R(\lambda, A)y_n(t) = \int_0^t e^{(t-\tau)A} R(\lambda, A) F_n(\tau, u_0(\tau))\, d\tau = \int_0^t e^{(t-\tau)A} f_n(\tau)\, d\tau$$

where $f_n : [0, +\infty) \to X$, $n \geq 0$, are defined as follows

$$f_n(\tau) = R(\lambda, A) F_n(\tau, u_0(\tau)).$$

As $R(\lambda, A)$ is continuous, so are f_n, $n \geq 0$. Moreover, $\operatorname{Im} R(\lambda, A) = D(A)$, therefore, for all $n \geq 0$ and $\tau \geq 0$,

$$\int_0^\tau f_n(\sigma)\, d\sigma = R(\lambda, A) \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma \in D(A).$$

Observe that the linear operator $AR(\lambda, A) : X \to X$ is bounded:

$$AR(\lambda, A) = (A - \lambda I + \lambda I) R(\lambda, A) = -I + \lambda R(\lambda, A).$$

Hence, for all $n \geq 0$ and $\tau \geq 0$,

$$A \int_0^\tau f_n(\sigma)\, d\sigma = AR(\lambda, A) \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma = \int_0^\tau AR(\lambda, A) F_n(\sigma, u_0(\sigma))\, d\sigma.$$

Therefore, the map

$$[0, +\infty) \ni \tau \mapsto A \int_0^\tau f_n(\sigma)\, d\sigma \in X$$

is differentiable, hence, in particular, continuous. Then Lemma 1.3.4 yields, for all $n \geq 0$,

$$R(\lambda, A)y_n(t) = \int_0^t e^{(t-\tau)A} f_n(\tau)\, d\tau = \int_0^t f_n(\tau)\, d\tau + \int_0^t e^{(t-\tau)A} A \int_0^\tau f_n(\sigma)\, d\sigma\, d\tau. \tag{1.29}$$

By assumption, (1.11) holds for all $\tau \geq 0$ and $u \in X$. Hence, Lemma 1.3.5 implies, for $\tau \in [0, t]$,

$$\int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma \to \int_0^\tau F_0(\sigma, u_0(\sigma))\, d\sigma \quad \text{as } n \to \infty. \tag{1.30}$$

Consequently,

$$\int_0^t f_n(\tau)\, d\tau \to \int_0^t f_0(\tau)\, d\tau \quad \text{as } n \to \infty. \tag{1.31}$$

Further, note that

$$\begin{aligned}
\int_0^t e^{(t-\tau)A} A \int_0^\tau f_n(\sigma)\, d\sigma\, d\tau &= \int_0^t e^{(t-\tau)A} A \int_0^\tau R(\lambda, A) F_n(\sigma, u_0(\sigma))\, d\sigma\, d\tau \\
&= \int_0^t e^{(t-\tau)A} \big(AR(\lambda, A)\big) \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma\, d\tau \\
&= \int_0^t \big(AR(\lambda, A)\big) e^{(t-\tau)A} \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma\, d\tau \\
&= \big(AR(\lambda, A)\big) \int_0^t e^{(t-\tau)A} \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma\, d\tau.
\end{aligned} \tag{1.32}$$

Moreover, in the light of (1.30), for any $\tau \in [0, t]$,

$$e^{(t-\tau)A} \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma \to e^{(t-\tau)A} \int_0^\tau F_0(\sigma, u_0(\sigma))\, d\sigma \quad \text{as } n \to \infty$$

and, for all $n \geq 1$ and $\tau \in [0,t]$,

$$\left\| e^{(t-\tau)A} \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma \right\| \leq \|e^{(t-\tau)A}\|_{\mathcal{L}(X)} \int_0^\tau \|F_n(\sigma, u_0(\sigma))\|\, d\sigma \leq tC_1C_2.$$

Therefore, due to the Dominated Convergence Theorem,

$$\int_0^t e^{(t-\tau)A} \int_0^\tau F_n(\sigma, u_0(\sigma))\, d\sigma\, d\tau \to \int_0^t e^{(t-\tau)A} \int_0^\tau F_0(\sigma, u_0(\sigma))\, d\sigma\, d\tau \quad \text{as } n \to \infty.$$

Together with (1.29), (1.31) and (1.32), this yields

$$R(\lambda, A)y_n(t) \to R(\lambda, A)y_0(t) \quad \text{as } n \to \infty.$$

This and (1.28) allow us to apply Lemma 1.3.2, which gives $y_n(t) \to y_0(t)$. Then it follows from (1.25) that $\gamma_n^{(2)}(t) \to 0$. As a result, we obtain (1.24).

Since $(\overline{u}_n)_{n\geq 1}$ is bounded, the sequence of functions $(\gamma_n^{(1)})_{n\geq 1}$ is uniformly bounded on $[0,T_0]$. Note that, for any $t \in [0,T_0]$ and $n \geq 0$,

$$\left\| \int_0^t e^{(t-\tau)A} F_n(\tau, u_0(\tau))\, d\tau \right\| \leq T_0C_1C_2.$$

Hence, the sequence of functions $(\gamma_n^{(2)})_{n\geq 1}$ is uniformly bounded on $[0,T_0]$. Therefore, by virtue of (1.24) and the Lebesgue Dominated Convergence Theorem,

$$\int_0^{T_0} \gamma_n(\tau)\, d\tau \to 0 \quad \text{as } n \to \infty.$$

Thus, (1.24) and (1.23) yield that $u_n(t) \to u_0(t)$ for any $t \in [0,T_0]$. As we assume that $(\overline{u}_n)_{n\geq 1}$ is bounded and F_n, $n \geq 0$, satisfy conditions (F1) and (F2) with common constants, Remark 1.2.2 implies that the mild solutions $u_n : [0,+\infty) \to X$, $n \geq 0$, of $P(A, F_n, \overline{u}_n)$ (see p. 4) are uniformly bounded on $[0,T_0]$. Consequently, the continuous functions

$$[0,+\infty) \ni t \mapsto F_n(t, u_n(t)) \in X, \quad n \geq 0,$$

are uniformly bounded on $[0,T_0]$. Since $u_n(t) \to u_0(t)$ for any $t \in [0,T_0]$, based on Proposition 1.2.1 (ii), we claim that $\left\{ u_n|_{[0,T_0]} \right\}_{n\geq 1}$ is relatively compact in $C([0,T_0],X)$. We shall prove that any subsequence $(u_{n_k})_{k\geq 1}$ contains a further subsequence which converges to u_0 in $C([0,T_0],X)$. Indeed, because $\left\{ u_n|_{[0,T_0]} \right\}_{n\geq 1}$ is relatively compact in $C([0,T_0],X)$, we may assume, extracting a subsequence if necessary, that $u_{n_k}|_{[0,T_0]} \to u$ in $C([0,T_0],X)$. On the other hand, $u_{n_k}(t) \to u_0(t)$ for all $t \in [0,T_0]$. This means that $u_0(t) = u(t)$ for all $t \in [0,T_0]$, hence $u_n \to u_0$ in $C([0,T_0],X)$. Consequently, since the interval $[0,T_0]$ was arbitrary, we obtain that $u_n(t) \to u_0(t)$ uniformly for t on bounded subsets of $[0,+\infty)$. The proof is completed. □

1.4 Continuity for parametrized equations and averaging principle

From the main result of the chapter – Theorem 1.3.1 – we derive the following continuity property for parametrized equations.

Theorem 1.4.1. *Assume that X is a separable Banach space and mappings $F_n : [0,+\infty)\times X \to X$, $n \geq 0$, are continuous and satisfy conditions* (F1) *and* (F2) *with constants independent of n. In addition, suppose that F_n, $n \geq 0$, are such that the property* (1.11) *holds, for any $t > 0$ and $u \in X$, $\{F_n(t,u) : t \in D, n \geq 1\}$ is relatively compact, for any bounded $D \subset [0,+\infty)$ and $u \in X$, $\alpha_n \to \alpha_0$ in $(0,+\infty)$ and $\overline{u}_n \to \overline{u}_0$ in X.*

If $u_n : [0,+\infty) \to X$, $n \geq 0$, are the mild solutions of $P(A, F_n(\cdot/\alpha_n, \cdot), \overline{u}_n)$ (see p. 4), then $u_n(t) \to u_0(t)$ uniformly for t on bounded subsets of $[0,+\infty)$.

Proof. Let us define the mappings $\widetilde{F}_n : [0,+\infty) \times X \to X$, $n \geq 0$, as follows: $\widetilde{F}_n(t,u) = F_n(t/\alpha_n, u)$. Obviously, $\widetilde{F}_n$, $n \geq 0$, are continuous and satisfy (F1) and (F2) with constants independent of n. Similarly, one gets that the set $\{\widetilde{F}_n(t,u)\}_{n\geq 1}$ is relatively compact for any $t \geq 0$ and $u \in X$, because, by assumption, $\{F_n(t,u) : t \in D, n \geq 1\}$ is relatively compact for any bounded set $D \subset [0,+\infty)$ and vector $u \in X$. Next, we shall prove that

$$\int_0^t \widetilde{F}_n(\tau,u)\,d\tau \to \int_0^t \widetilde{F}_0(\tau,u)\,d\tau \quad \text{as } n \to \infty \tag{1.33}$$

for any $t > 0$ and $u \in X$. Indeed, note that

$$\int_0^t \widetilde{F}_n(\tau,u)\,d\tau = \alpha_n \int_0^{t/\alpha_n} F_n(\tau,u)\,d\tau \quad \text{for } n \geq 0.$$

Directly from the assumption we obtain that

$$\int_0^{t/\alpha_0} F_n(\tau,u)\,d\tau \to \int_0^{t/\alpha_0} F_0(\tau,u)\,d\tau \quad \text{as } n \to \infty. \tag{1.34}$$

On the other hand, by the Lipschitz and boundedness at zero conditions, there exists a constant $C > 0$ such that $\|F_n(\tau,u)\| \leq C$ for all $\tau \in [0,+\infty)$ and $n \geq 1$. Therefore,

$$\left\| \int_0^{t/\alpha_n} F_n(\tau,u)\,d\tau - \int_0^{t/\alpha_0} F_n(\tau,u)\,d\tau \right\| = \left\| \int_{t/\alpha_0}^{t/\alpha_n} F_n(\tau,u)\,d\tau \right\| \leq |t/\alpha_n - t/\alpha_0| C \to 0 \text{ as } n \to \infty.$$

From this and (1.34) we deduce (1.33).

Thus, we can apply Theorem 1.3.1 to get that $u_n(t) \to u_0(t)$, uniformly for t from bounded subsets of $[0,+\infty)$. This finishes the proof. □

Now, we apply the main result of the chapter to get the averaging principle.

Theorem 1.4.2. *Suppose that X is a separable Banach space, $F_n : [0,+\infty) \times X \to X$, $n \geq 1$, are continuous and satisfy conditions* (F1) *and* (F2) *with constants independent of $n \geq 1$, and*

(F3) *F_n are T-periodic, that is, there exists $T > 0$ such that*

$$F_n(t+T,u) = F_n(t,u)$$

for all $t \geq 0$, $u \in X$ and $n \geq 1$.

Furthermore, assume that $\overline{u}_n \to \overline{u}_0$ in X, $(\varepsilon_n)_{n\geq 1}$ in $(0,+\infty)$ is such that $\varepsilon_n \to 0$, the set $\{F_n(t,u) : t \in [0,T], n \geq 1\}$ is relatively compact in X, for all $u \in X$, and there exists the mapping $\widehat{F} : X \to X$ given by

$$\widehat{F}(u) = \lim_{n\to\infty} \widehat{F}_n(u) \quad \textit{for any } u \in X$$

where the mappings $\widehat{F}_n : X \to X$, $n \geq 1$, are defined as

$$\widehat{F}_n(u) = \frac{1}{T}\int_0^T F_n(t,u)\,dt \quad \textit{for any } u \in X.$$

If, for all $n \geq 1$, $u_n : [0,+\infty) \to X$ is the mild solution of

$$\begin{cases} \dot{u}(t) = Au(t) + F_n(t/\varepsilon_n, u(t)), & t > 0, \\ u(0) = \overline{u}_n \end{cases} \tag{1.35}$$

and $\widehat{u} : [0,+\infty) \to X$ is the mild solution of

$$\begin{cases} \dot{u}(t) = Au(t) + \widehat{F}(u(t)), & t > 0, \\ u(0) = \overline{u}_0, \end{cases} \tag{1.36}$$

then $u_n(t) \to \widehat{u}(t)$ uniformly for t on bounded subsets of $[0,+\infty)$.

Proof. We shall apply Theorem 1.3.1. To this end, let us define the mappings $\widetilde{F}_n : [0,+\infty)\times X \to X$, $n \geq 1$, $\widetilde{F}_n(t,u) = F_n(t/\varepsilon_n, u)$ and $\widetilde{F}_0 : [0,+\infty)\times X \to X$, $\widetilde{F}_0(t,u) = \widehat{F}(u)$. We see that $\widetilde{F}_n$, $n \geq 1$, are continuous and satisfy (F1) and (F2) with the same constants as F_n, $n \geq 1$.

Since $\widehat{F}$ is t–independent, it is only necessary to prove that $\widehat{F}$ meets the condition (F1). In fact, we take any u_1, $u_2 \in X$ and obtain

$$\begin{aligned}\left\| \frac{1}{T}\int_0^T F_n(t,u_1)\,dt - \frac{1}{T}\int_0^T F_n(t,u_2)\,dt \right\| &\leq \frac{1}{T}\int_0^T \|F_n(t,u_1) - F_n(t,u_2)\|\,dt \\ &\leq \frac{1}{T}\int_0^T L\|u_1-u_2\|\,dt = L\|u_1-u_2\| \quad \text{for } n \geq 1.\end{aligned}$$

Hence, as $n \to \infty$, we arrive at

$$\|\widehat{F}(u_1) - \widehat{F}(u_2)\| \leq L\|u_1-u_2\|,$$

i.e. $\widehat{F}$ fulfills the Lipschitz condition with the same constant as mappings F_n, $n \geq 1$. Further, by the T-periodicity, we have, for any $t \in [0,+\infty)$ and $u \in X$,

$$\{\widetilde{F}_n(t,u)\}_{n\geq 1} \subset \{F_n(t,u) : t \in [0,T], n \geq 1\},$$

thus, the set $\{\widetilde{F}_n(t,u)\}_{n\geq 1}$ is relatively compact.

As, for each $n \geq 1$, the map F_n is T-periodic, observe that, changing variables in the integrals, for any $t > 0$, $u \in X$ and $n \geq 1$,

$$\int_0^t \widetilde{F}_n(\tau,u)\,d\tau = \int_0^t F_n(\tau/\varepsilon_n,u)\,d\tau = \frac{t}{t/\varepsilon_n}\int_0^{t/\varepsilon_n} F_n(\tau,u)\,d\tau$$

and, denoting $t_n = t/\varepsilon_n$, we have further ($[a]$ denotes the integer part of $a \in \mathbb{R}$)

$$\begin{aligned}\frac{t}{t_n}\int_0^{t_n} F_n(\tau,u)\,d\tau &= \frac{t}{t_n/T}\,\frac{1}{T}\Bigg(\sum_{k=0}^{[t_n/T]-1}\int_{kT}^{(k+1)T} F_n(\tau,u)\,d\tau + \int_{[t_n/T]T}^{t_n} F_n(\tau,u)\,d\tau\Bigg) \\ &= \frac{t}{t_n/T}\,\frac{1}{T}\Bigg(\sum_{k=0}^{[t_n/T]-1}\int_{kT}^{(k+1)T} F_n(\tau-kT,u)\,d\tau + \int_{[t_n/T]T}^{t_n} F_n(\tau-[t_n/T]T,u)\,d\tau\Bigg) \\ &= \frac{t}{t_n/T}\,\frac{1}{T}\Bigg(\sum_{k=0}^{[t_n/T]-1}\int_0^T F_n(\tau,u)\,d\tau + \int_0^{t_n-[t_n/T]T} F_n(\tau,u)\,d\tau\Bigg) \\ &= t\,\frac{[t_n/T]}{t_n/T}\,\frac{1}{T}\int_0^T F_n(\tau,u)\,d\tau + \frac{t}{t_n}\int_0^{t_n-[t_n/T]T} F_n(\tau,u)\,d\tau.\end{aligned}$$

Since F_n, $n \geq 1$, satisfy conditions (F1) and (F2) with common constants, they satisfy the sublinear growth condition with a common constant (see (1.6)). Hence, we obtain

$$\left\| \frac{t}{t_n}\int_0^{t_n-[t_n/T]T} F_n(\tau,u)\,d\tau \right\| \leq \frac{t}{t_n}\int_0^T \|F_n(\tau,u)\|\,d\tau \to 0 \quad \text{as } n \to \infty.$$

On the other hand, $[t_n/T]/(t_n/T) \to 1$ and, by assumption,

$$t\,\frac{[t_n/T]}{t_n/T}\,\frac{1}{T}\int_0^T F_n(\tau,u)\,d\tau = t\,\frac{[t_n/T]}{t_n/T}\,\widehat{F}_n(u) \to t\,\widehat{F}(u) = \int_0^t \widetilde{F}_0(\tau,u)\,d\tau \ \text{ as } \ n \to \infty.$$

Thus, all the assumptions of Theorem 1.3.1 are satisfied and its application yields the assertion. □

1.5 Translation along trajectories operator

Given a Banach space $(X, \|\cdot\|)$ and a compact metric space (M, d), let $\{A(\mu) : D(A(\mu)) \subset X \to X\}_{\mu \in M}$ be the family of linear operators satisfying

(A1) for any $\mu \in M$, $A(\mu)$ is the generator of a C_0-semigroup $\{e^{tA(\mu)}\}_{t \geq 0}$ such that

$$\|e^{tA(\mu)}\|_{\mathcal{L}(X)} \leq Ke^{\omega t}$$

for all $t \geq 0$ and some numbers $K \geq 1$ and $\omega \in \mathbb{R}$;

(A2) for any sequence $\mu_n \to \mu_0$ in M one has, for any $u \in X$ and $\lambda > \omega$([2]),

$$R(\lambda, A(\mu_n))u \xrightarrow{n \to \infty} R(\lambda, A(\mu_0))u.$$

Suppose that a continuous mapping $H : [0, +\infty) \times X \times M \to X$ is such that

(H1) H fulfills the (uniform) Lipschitz condition, that is, there exists $L > 0$ such that, for all $t \geq 0$, $\mu \in M$ and u_1, $u_2 \in X$,

$$\|H(t, u_1, \mu) - H(t, u_2, \mu)\| \leq L\|u_1 - u_2\|;$$

(H2) H has a (uniform) sublinear growth, i.e., there exists $C > 0$ such that $\|H(t, u, \mu)\| \leq C(1 + \|u\|)$, for all $t \geq 0$, $u \in X$ and $\mu \in M$.

Consider the family of differential equations

$$\dot{u}(t) = A(\mu)u(t) + H(t, u(t), \mu), \quad t > 0, \ \mu \in M. \tag{1.37}$$

Now, we associate with the family of equations (1.37) the *translation along trajectories operator* (by time $t > 0$) $\Phi_t : X \times M \to X$, also known as the *Poincaré operator*. To this end, recall that, for any $\overline{u} \in X$ and $\mu \in M$, there exists the unique global mild solution $u(\cdot, \overline{u}, \mu) : [0, +\infty) \to X$ of (1.37) with $u(0) = \overline{u}$ – see Section 1.2. Then Φ_t is defined as follows:

$$\Phi_t(\overline{u}, \mu) = u(t, \overline{u}, \mu).$$

The following theorem concerns continuity and compactness properties of the translation along trajectories operator.

Theorem 1.5.1. [17, Prop. 4.1] *Let the family of linear operators* $\{A(\mu) : D(A(\mu)) \subset X \to X\}_{\mu \in M}$ *satisfy* (A1) *and* (A2), *and the mapping* $H : [0, +\infty) \times X \times M \to X$ *be continuous and satisfy* (H1) *and* (H2).

(i) Assume that $\overline{u}_n \to \overline{u}_0$ *in* X *and* $\mu_n \to \mu_0$ *in* M, *and let* $u_n : [0, +\infty) \to X$, $n \geq 0$, *be the mild solutions of* (1.37) *with* $u_n(0) = \overline{u}_n$ *and* $\mu = \mu_n$. *Then* $u_n(t) \to u_0(t)$ *uniformly for* t *on bounded subsets of* $[0, +\infty)$.

(ii) Let χ *be the Hausdorff measure of non-compactness in* X *(see Appendix A.3), and assume additionally that* X *is separable. If there exists a number* $\rho > 0$ *such that*

$$\chi(\{e^{tA(\mu)}u : u \in U, \mu \in M\}) \leq e^{-\rho t}\chi(U)$$

for any $t > 0$ *and bounded* $U \subset X$, *and there exists a number* $k \geq 0$ *such that*

$$\chi(H(D \times U \times M)) \leq k\chi(U)$$

for any bounded sets $D \subset [0, +\infty)$ *and* $U \subset X$, *then*

$$\chi(\Phi_t(U \times M)) \leq e^{-(\rho - k)t}\chi(U)$$

for any $t > 0$ *and any bounded* $U \subset X$.

Remark 1.5.2. By the point (i) of Theorem 1.5.1, the translations along trajectories operator $\Phi_t : X \times M \to X$ is continuous and, by the point (ii), if $\rho > k$, then Φ_t is the so-called k-set contraction – see Section A.4. □

([2])In view of the Hille-Yosida Theorem (Theorem 1.1.5), (A1) yields $\rho(A(\mu)) \subset (\omega, +\infty)$ for any $\mu \in M$.

Chapter 2

Linear damped wave equation

This chapter is dedicated to the linear damped wave equation:

$$u_{tt} + \beta u_t = \Delta u - V(x)u, \quad t > 0.$$

In Section 2.1, we recall that the Schrödinger operator with a Kato-Rellich type potential is well-defined, closed, and self-adjoint in $L^2(\mathbb{R}^N)$, and we discuss its spectrum. In Section 2.2, we analyze the spectrum of the damped wave operator and prove that it generates a C_0-semigroup. In Section 2.3, we define an equivalent norm on the space $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, and we show that the semigroup generated by the damped wave operator is a k-set contraction with respect to the Hausdorff measure of non-compactness on $\mathbb{X}$, equipped with this norm. Finally, in Section 2.4, we provide a formula for the topological fixed-point index of this semigroup.

2.1 Schrödinger operator with a Kato-Rellich type potential

In this section we investigate the Schrödinger operator $-\Delta + V$. To this end, we define linear operator $A : D(A) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$Au = -\Delta u + \boldsymbol{V}u \quad \text{for} \quad u \in D(A) = H^2(\mathbb{R}^N) \tag{2.1}$$

where the linear operator $-\Delta : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined by use of the second order weak derivatives:

$$-\Delta u = -\sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2} \quad \text{for} \quad u \in H^2(\mathbb{R}^N). \tag{2.2}$$

It is well-known that $-\Delta$ is closed and self-adjoint, see [36, Example 8.4]. The linear operator $\boldsymbol{V} : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined as follows

$$[\boldsymbol{V}u](x) = V(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N, \tag{2.3}$$

where V is a Kato-Rellich type potential (see (5a) and (5b)).

Example 2.1.1. Consider function $V : \mathbb{R}^N \setminus \{x_0\} \to \mathbb{R}$ defined by

$$V(x) = \frac{a}{|x - x_0|^\gamma} + b \tag{2.4}$$

where a, b are real numbers, $x_0 \in \mathbb{R}^N$ and $\gamma > 0$ is such that

$$\begin{cases} \gamma \in (0, N/2) & \text{if } N \in \{1, 2, 3, 4\}, \\ \gamma \in (0, 2) & \text{if } N \geq 5. \end{cases}$$

We show that then V is a Kato-Rellich type potential. In particular, the Coulomb potential $V_C : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, defined by $V_C(x) = a/|x|$, $a \in \mathbb{R}$, belongs to this class.

To this end, we decompose V as follows:

$$V(x) = V_\infty(x) + V_0(x)$$

where

$$V_\infty(x) = \frac{a}{|x-x_0|^\gamma}\,\mathbf{1}_{\mathbb{R}^N\setminus B(x_0,1)}(x) + b \quad\text{and}\quad V_0(x) = \frac{a}{|x-x_0|^\gamma}\,\mathbf{1}_{B(x_0,1)}(x). \tag{2.5}$$

We note immediately that $V_\infty \in L^\infty(\mathbb{R}^N)$ as $|V_\infty(x)| \le |a| + |b|$ for almost every $x \in \mathbb{R}^N$. Additionally, there holds $\varrho(V_\infty) = \widehat{\varrho}(V_\infty) = b$, where the number $\varrho(V_\infty)$ is called asymptotic bottom (see (6)), and the number $\widehat{\varrho}(V_\infty)$ is given by (14). Indeed, we have, for all $R \ge 1$,

$$\operatorname*{ess\,inf}_{\mathbb{R}^N\setminus D(0,R)} V_\infty = \begin{cases} b & \text{if} \quad a \ge 0, \\ \dfrac{a}{R^\gamma} + b & \text{if} \quad a < 0 \end{cases} \quad\text{and}\quad \operatorname*{ess\,sup}_{\mathbb{R}^N\setminus B(0,R)} V_\infty = \begin{cases} \dfrac{a}{R^\gamma} + b & \text{if} \quad a \ge 0, \\ b & \text{if} \quad a < 0 \end{cases}$$

and hence the result. Moreover, we have, for all $p \in (0, N/\gamma)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |V_0(x)|^p\,dx &= \int_{\mathbb{R}^N} \frac{|a|^p}{|x-x_0|^{\gamma p}}\,\mathbf{1}_{B(x_0,1)}(x)\,dx = |a|^p \int_{B(x_0,1)} \frac{1}{|x-x_0|^{\gamma p}}\,dx \\
&= |a|^p \int_{B(0,1)} \frac{1}{|x|^{\gamma p}}\,dx = |a|^p \int_0^1 \int_{S(0,r)} \frac{1}{|x|^{\gamma p}}\,dS(x)\,dr \\
&= |a|^p \int_0^1 \int_{S(0,r)} \frac{1}{r^{\gamma p}}\,dS(x)\,dr = |a|^p \int_0^1 \frac{|S(0,r)|}{r^{\gamma p}}\,dr \\
&= |a|^p \int_0^1 \frac{r^{N-1}|S(0,1)|}{r^{\gamma p}}\,dr = \frac{|a|^p\,|S(0,1)|}{N-\gamma p}
\end{aligned} \tag{2.6}$$

where $S(0,r)$ is the $N-1$-dimensional sphere of radius r centered at the origin, and $|S(0,r)|$ denotes its surface area. If $N \in \{1,2,3,4\}$, then, by assumption, $\gamma \in (0, N/2)$ and hence $2 < N/\gamma$. Therefore, if we take arbitrary $2 < p < N/\gamma$, then, by (2.6), $V_0 \in L^p(\mathbb{R}^N)$. Subsequently, if $N \ge 5$, then, by assumption, $\gamma \in (0,2)$ and hence $N/2 < N/\gamma$. Therefore, if we take arbitrary $N/2 \le p < N/\gamma$, then, by (2.6), $V_0 \in L^p(\mathbb{R}^N)$. Summarizing, for any $N \ge 1$, $V_0 \in L^p(\mathbb{R}^N)$, where exponent p is as in (5b). This shows that V is a Kato-Rellich type potential.

If we assume that the exponent $\gamma > 0$ satisfies

$$\begin{cases} \gamma \in (0, 1/2) & \text{if } N = 1, \\ \gamma \in (0,1) & \text{if } N \ge 2 \end{cases}$$

then V is a Kato-Rellich type function. To show this, it suffices to decompose the function V as in (2.5) and use the estimate (2.6) to obtain that $V_0 \in L^p(\mathbb{R}^N)$, with the exponent $p \ge 2$ as in condition (8b). □

Let the linear operator $\boldsymbol{V}_\infty : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be given by

$$[\boldsymbol{V}_\infty u](x) = V_\infty(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N,$$

and $A_\infty : D(A_\infty) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$A_\infty u = -\Delta u + \boldsymbol{V}_\infty u \quad \text{for} \quad u \in D(A_\infty) = H^2(\mathbb{R}^N). \tag{2.7}$$

We also introduce the linear operator $\boldsymbol{V}_0 : \mathcal{D}(\boldsymbol{V}_0) \to L^2(\mathbb{R}^N)$,

$$[\boldsymbol{V}_0 u](x) = V_0(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N$$

where the space $\mathcal{D}(\boldsymbol{V}_0)$ is given by

$$\mathcal{D}(\boldsymbol{V}_0) = \begin{cases} L^\infty(\mathbb{R}^N) & \text{if } p = 2, \\ L^{2p/(p-2)}(\mathbb{R}^N) & \text{if } p > 2. \end{cases} \tag{2.8}$$

Firstly, we shall justify that the operators A and A_∞ are well-defined, closed and self-adjoint, and then provide some results concerning relative compactness (Proposition 2.1.10). Secondly, we use this result do determine the spectrum of A (Proposition 2.1.12). Since in Chapter 3 (Corollary 3.1.1 and the proof of Lemma 3.3.2 – see p. 60) we will investigate properties of the non-negative Kato-Rellich type functions (see (8a), (8b), and (9)), we consider now a general linear operator $\boldsymbol{W} : H^k(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$,

$$[\boldsymbol{W}u](x) = W(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N$$

where $k \geq 1$ is an integer,

$$W = W_\infty + W_0 \quad \text{for} \quad W_\infty \in L^\infty(\mathbb{R}^N),\ W_0 \in L^p(\mathbb{R}^N), \tag{2.9a}$$

and

$$\begin{cases} p \geq 2 & \text{if } N \in \{1, \dots, 2k-1\}, \\ p > 2 & \text{if } N = 2k, \\ p \geq N/k & \text{if } N \geq 2k+1. \end{cases} \tag{2.9b}$$

Moreover, we define the linear operators $\boldsymbol{W}_\infty : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$,

$$[\boldsymbol{W}_\infty u](x) = W_\infty(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N,$$

and $\boldsymbol{W}_0 : \mathcal{D}(\boldsymbol{W}_0) \to L^2(\mathbb{R}^N)$ as

$$[\boldsymbol{W}_0 u](x) = W_0(x)u(x) \quad \text{for} \quad x \in \mathbb{R}^N$$

where $\mathcal{D}(\boldsymbol{W}_0) = \mathcal{D}(\boldsymbol{V}_0)$ (see (2.8)). Observe that, for $k = 2$, W is a Kato-Rellich type potential and, for $k = 1$, W is a Kato-Rellich type function.

Remark 2.1.2. By the Sobolev theorem (Theorem A.2.4), for any $k \geq 1$, the following embeddings are continuous

$$\begin{array}{lll} H^k(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) & \text{for } r \in [2, 2N/(N-2k)] & \text{if } N \geq 2k+1, \\ H^k(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) & \text{for } r \in [2, +\infty) & \text{if } N = 2k, \\ H^k(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) & \text{for } r \in [2, +\infty] & \text{if } N \in \{1, \dots, 2k-1\}. \end{array}$$

The above embeddings are still true if we replace $\mathbb{R}^N$ by an open $U \subset \mathbb{R}^N$ of class C^1 with bounded boundary(1). □

Lemma 2.1.3. *Let $k \geq 1$ and $p \geq 2$ be as in* (2.9b)*. Then the linear operators $\boldsymbol{W}_\infty$, $\boldsymbol{W}_0$ are well-defined and bounded. Moreover, the space $H^k(\mathbb{R}^N)$ is continuously embedded into $\mathcal{D}(\boldsymbol{W}_0)$, and the linear operator $\boldsymbol{W} = \boldsymbol{W}_\infty + \boldsymbol{W}_0$ is well-defined. In addition, operator $\boldsymbol{W}$ considered as a linear operator from the space $H^k(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ is bounded, and, for any $u \in H^k(\mathbb{R}^N)$,*

$$\|\boldsymbol{W}u\|_{L^2} \leq \|W_\infty\|_{L^\infty}\|u\|_{L^2} + C\|W_0\|_{L^p}\|u\|_{H^k}$$

where $C = C(N,p) > 0$ is a constant.

(1)We say also that U is a set with boundary of class C^1. For the definition of sets of class C^1 see [9, p. 272].

Proof. We see that, for $u \in L^2(\mathbb{R}^N)$, $\|W_\infty u\|_{L^2} \le \|W_\infty\|_{L^\infty}\|u\|_{L^2}$, hence $\boldsymbol{W}_\infty u \in L^2(\mathbb{R}^N)$ and furthermore, $\boldsymbol{W}_\infty$ is bounded.

If $p = 2$, then we have $\|W_0 u\|_{L^2} \le \|W_0\|_{L^2}\|u\|_{L^\infty}$, i.e. $\boldsymbol{W}_0$ is well-defined and bounded. If $p > 2$, then, for any $u \in L^{2p/(p-2)}(\mathbb{R}^N)$, we apply the Hölder inequality to obtain

$$\left(\int_{\mathbb{R}^N} |W_0(x)|^2\,|u(x)|^2\,dx\right)^{1/2} \le \|W_0\|_{L^p}\,\|u\|_{L^{2p/(p-2)}}.$$

This shows that $\boldsymbol{W}_0$ is well-defined and bounded also for $p > 2$.

In view of Remark 2.1.2 the space $H^k(\mathbb{R}^N)$ is continuously embedded into $\mathcal{D}(\boldsymbol{W}_0)$, for all $k \ge 1$ and $p \ge 2$ as in (2.9b). Hence, we get that $\boldsymbol{W}$ is well-defined and bounded. □

Remark 2.1.4. From Remark 2.1.2 we deduce that Lemma 2.1.3 holds if $\mathbb{R}^N$ is replaced by an open set $U \subset \mathbb{R}^N$ of class C^1 with bounded boundary. □

The following proposition concerns compactness of the operator $\boldsymbol{W}_0$.

Proposition 2.1.5. *Suppose that $k \ge 1$ and $p \ge 2$ are as in* (2.9b)*. Then the restricted linear operator $\boldsymbol{W}_0|_{H^k(\mathbb{R}^N)}$ is compact, that is, the set $\boldsymbol{W}_0(U)$ is relatively compact in $L^2(\mathbb{R}^N)$, for any bounded $U \subset H^k(\mathbb{R}^N)$.*

The proof relies on several facts.

Lemma 2.1.6. *Assume that $U \subset \mathbb{R}^N$ is an open set of class C^1 with bounded boundary and $W_0 \in L^p(\mathbb{R}^N)$ with $k \ge 1$ and $p > 2$ as in* (2.9b)*. Then, for any $u \in H^k(\mathbb{R}^N)$,*

$$\|W_0\, u\|_{L^2(U)} \le C^\theta\; \|W_0\|_{L^p(U)}\; \|u\|^\theta_{H^k(U)}\; \|u\|^{1-\theta}_{L^2(U)} \tag{2.10}$$

where $C = C(k,p,r,N,U) > 0$ and

$$\begin{cases} r > 2p/(p-2), & \theta = 2r/p(r-2), \quad \theta \in (0,1) \quad \text{if} \quad N \in \{1,\dots,2k\} \\ r = 2N/(N-2k), & \theta = (N/k)/p, \quad \theta \in (0,1] \quad \text{if} \quad N \ge 2k+1. \end{cases}$$

Proof. It follows from the Hölder inequality that

$$\|W_0\, u\|_{L^2(U)} \le \|W_0\|_{L^p(U)}\; \|u\|_{L^{2p/(p-2)}(U)}. \tag{2.11}$$

We take $r \ge 2p/(p-2)$ and, by the interpolation inequality (cf. Proposition A.2.2 (i)), we get

$$\|u\|_{L^{2p/(p-2)}(U)} \le \|u\|^{1-\theta}_{L^2(U)}\; \|u\|^\theta_{L^r(U)} \tag{2.12}$$

with $\theta = 2r/p(r-2)$. Note that $\theta \in (0,1]$.

Let $N \in \{1,\dots,2k\}$ and fix $r > 2p/(p-2)$. Based on Remark 2.1.2, one has $H^k(U) \hookrightarrow L^r(U)$ and then (2.11) combined with (2.12) yields the desired inequality. In addition, $\theta \in (0,1)$, because $r > 2p/(p-2)$.

Assume that $N \ge 2k+1$ and let $r = 2N/(N-2k)$. In the light of Remark 2.1.2, there holds $H^k(U) \hookrightarrow L^r(U)$ and, since $r \ge 2p/(p-2)$, from (2.11) and (2.12) we deduce (2.10). □

It will be important in the proof of Proposition 2.1.5 that $\|u\|_{L^2}$ in the inequality (2.10) is raised to the positive power. However, in the case $N \ge 2k+1$ and $p = N/k$ we have $1-\theta = 1-(N/k)/p = 0$, therefore we need two lemmata below.

Lemma 2.1.7. *Suppose that $k \ge 1$ and $N \ge 2k+1$, $W_0 \in L^{N/k}(\mathbb{R}^N)$, $U \subset \mathbb{R}^N$ is an open set of class C^1 with bounded boundary and finite measure, and $b \in (0,1)$. Then, for all $u \in H^k(\mathbb{R}^N)$,*

$$\|W_0\, u\|_{L^2(U)} \le C\; \big\||W_0| - |W_0|^b\big\|_{L^{N/k}(U)}\; \|u\|_{H^k(U)} + C^b\; \|W_0\|^b_{L^{N/k}(U)}\; \|u\|^b_{H^k(U)}\; \|u\|^{1-b}_{L^2(U)}$$

where $C = C(N,k) > 0$ is a constant.

Proof. Observe that $|W_0|^b \in L^{(N/k)/b}(U)$, because $W_0 \in L^{N/k}(U)$. The set U has a finite measure and $(N/k)/b > N/k$, hence $|W_0|^b \in L^{N/k}(U)$. On the other hand, $u|_U \in H^k(U)$ and, by Remark 2.1.4, $|W_0|^b\, u \in L^2(U)$. We get the following inequality

$$\|W_0\, u\|_{L^2(U)} \leq \big\||W_0|\, u - |W_0|^b\, u\big\|_{L^2(U)} + \||W_0|^b\, u\|_{L^2(U)}. \tag{2.13}$$

Because U is an open set of class C^1 with bounded boundary, using Lemma 2.1.6 twice, firstly for $p = N/k$, we obtain

$$\big\|\big(|W_0| - |W_0|^b\big)\, u\big\|_{L^2(U)} \leq C\, \big\||W_0| - |W_0|^b\big\|_{L^{N/k}(U)}\, \|u\|_{H^k(U)},$$

and secondly, for $p = (N/k)/b$, we obtain

$$\||W_0|^b\, u\|_{L^2(U)} \leq C^b\, \|W_0\|^b_{L^{N/k}(U)}\, \|u\|^b_{H^k(U)}\, \|u\|^{1-b}_{L^2(U)}$$

where $C = C(N,k) > 0$. Combining these inequalities with (2.13) we get the desired inequality. □

Lemma 2.1.8. *Let $f \in L^p(U)$ with $p \in [1,+\infty)$, where $U \subset \mathbb{R}^N$ is a set of finite measure. Then for any $b \in (0,1)$, we have*

$$\lim_{b\to 1^-} \big\||f| - |f|^b\big\|_{L^p(U)} = 0.$$

Proof. Note that, for almost every $x \in U$,

$$\big||f(x)| - |f(x)|^b\big|^p \leq \big(|f(x)| + |f(x)|^b\big)^p \leq c(x)$$

where the function $c : U \to \mathbb{R}$ is given by

$$c(x) = \begin{cases} 2^p\, |f(x)|^p & \text{if } |f(x)| \geq 1, \\ 2^p & \text{if } |f(x)| < 1. \end{cases}$$

The function c is integrable, because $f \in L^p(U)$ and the measure of the set U is finite. Then the lemma follows by the application of the Lebesgue Dominated Convergence Theorem. □

Proof of Proposition 2.1.5. Let $(u_n)_{n\geq 1}$ be a bounded sequence in $H^k(\mathbb{R}^N)$. We shall prove that the set $\{\boldsymbol{W}_0 u_n\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. The Hölder inequality and the continuity of the embedding $H^k(\mathbb{R}^N) \hookrightarrow \mathcal{D}(\boldsymbol{W}_0)$ (see Lemma 2.1.3) yield, for any $r > 0$ and p as in (2.9b),

$$\|W_0 u_n\|_{L^2(\mathbb{R}^N\setminus B(0,r))} \leq C\, \|W_0\|_{L^p(\mathbb{R}^N\setminus B(0,r))}\, \|u_n\|_{H^k}, \quad n \geq 1, \tag{2.14}$$

where $C(N,p) > 0$ is a constant. In view of the Lebesgue Dominated Convergence Theorem, we get the *tail convergence*:

$$\lim_{r\to+\infty} \|W_0\|_{L^p(\mathbb{R}^N\setminus B(0,r))} = 0. \tag{2.15}$$

By assumption, the sequence $(u_n)_{n\geq 1}$ is bounded in $H^k(\mathbb{R}^N)$. Hence, by (2.14) and (2.15), for a given $\varepsilon > 0$ there exists $r > 0$ such that

$$\{W_0 u_n\}_{n\geq 1} \subset \{W_0 u_n \mathbf{1}_{B(0,r)}\}_{n\geq 1} + \{W_0 u_n \mathbf{1}_{\mathbb{R}^N\setminus B(0,r)}\}_{n\geq 1} \subset \{W_0 u_n \mathbf{1}_{B(0,r)}\}_{n\geq 1} + B_{L^2}(0,\varepsilon). \tag{2.16}$$

We claim that the set $\{W_0 u_n \mathbf{1}_{B(0,r)}\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Indeed, let us take a subsequence $(W_0 u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$. By the Rellich-Kondrachov Theorem (Theorem A.2.5), the set $\{u_{n_j} \mathbf{1}_{B(0,r)}\}_{j\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Hence, passing to a subsequence if necessary, $(u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$.

Assume that in (2.9b) we have $p > 2$ and $p \neq N/k$. By Lemma 2.1.6, there exist a constant $C > 0$ and $\theta \in (0, 1]$ such that

$$\|W_0 u_{n_j} - W_0 u_{n_l}\|_{L^2(B(0,r))} \leq C^\theta \, \|W_0\|_{L^p(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^\theta_{H^k(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^{1-\theta}_{L^2(B(0,r))}, \quad j, l \geq 1.$$

If either $N \in \{1, \ldots, 2k\}$ and $p > 2$, or $N \geq 2k + 1$ and $p > N/k$, then we have $\theta \in (0, 1)$, hence the above estimate shows that $(W_0 u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$.

In the case $N \in \{1, \ldots, 2k - 1\}$ and $p = 2$, using the Rellich-Kondrachov Theorem once again, we can assume that $(u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^N)$. Therefore, we have

$$\|W_0 u_{n_j} - W_0 u_{n_l}\|_{L^2(B(0,r))} \leq \|W_0\|_{L^2(B(0,r))} \, \|u_{n_j} - u_{n_l}\|_{L^\infty(B(0,r))} \to 0 \quad \text{as} \quad j, l \to \infty.$$

Assume that $N \geq 2k+1$ and $p = N/k$. We take some $\eta > 0$. By Lemma 2.1.7, there exists $C > 0$ such that, for any $b \in (0, 1)$ and all j, $l \geq 1$,

$$\begin{aligned} &\|W_0 u_{n_j} - W_0 u_{n_l}\|_{L^2(B(0,r))} \\ &\qquad \leq C \, \big\| |W_0| - |W_0|^b \big\|_{L^{N/k}(B(0,r))} \, \|u_{n_j} - u_{n_l}\|_{H^k(B(0,r))} \\ &\qquad\quad + C^b \, \|W_0\|^b_{L^{N/k}(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^b_{H^k(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^{1-b}_{L^2(B(0,r))}. \end{aligned} \tag{2.17}$$

Since $(u_n)_{n\geq 1}$ is bounded in $H^k(\mathbb{R}^N)$, based on Lemma 2.1.8, we take $b \in (0, 1)$ such that

$$C \, \big\| |W_0| - |W_0|^b \big\|_{L^{N/k}(B(0,r))} \, \|u_{n_j} - u_{n_l}\|_{H^k(B(0,r))} < \eta/2 \;\text{ for all } j, l \geq 1. \tag{2.18}$$

Because $(u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, there exists an integer $j_0 \geq 1$ such that, for the chosen $b \in (0, 1)$,

$$C^b \, \|W_0\|^b_{L^{N/k}(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^b_{H^k(B(0,r))} \, \|u_{n_j} - u_{n_l}\|^{1-b}_{L^2(B(0,r))} < \eta/2 \;\text{ for all } j, l \geq j_0.$$

Now, taking into account the inequality above, (2.17) and (2.18), we get

$$\|W_0 u_{n_j} - W_0 u_{n_l}\|_{L^2(B(0,r))} < \eta \;\text{ for all } j, l \geq j_0.$$

Considering that η is arbitrary, this shows that $(W_0 u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$ when $N \geq 2k + 1$ and $p = N/k$.

Since $(W_0 u_{n_j} \mathbf{1}_{B(0,r)})_{j\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$, it follows that for any p and N as in (2.9b), any subsequence of $(W_0 u_n \mathbf{1}_{B(0,r)})_{n\geq 1}$ contains a further subsequence that converges in $L^2(\mathbb{R}^N)$. This means that the set $\{W_0 u_n \mathbf{1}_{B(0,r)}\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$, as claimed. Consequently, the set $\{W_0 u_n \mathbf{1}_{B(0,r)}\}_{n\geq 1}$ is totally bounded in $L^2(\mathbb{R}^N)$. As $\varepsilon > 0$ is arbitrary, from (2.16) we deduce that $\{W_0 u_n\}_{n\geq 1}$ is totally bounded in $L^2(\mathbb{R}^N)$. Given that $L^2(\mathbb{R}^N)$ is a complete metric space, the total boundedness of a set is equivalent to its relative compactness. Thus, $\{W_0 u_n\}_{n\geq 1} = \{\boldsymbol{W}_0 u_n\}_{n\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. This finishes the proof. □

Consequence of Lemma 2.1.3 and Proposition 2.1.5 is the following result to the operators $\boldsymbol{V}_\infty$, $\boldsymbol{V}_0$, and $\boldsymbol{V}$.

Corollary 2.1.9. *Suppose that $p \geq 2$ is as in* (5b).

(i) The linear operators $\boldsymbol{V}_\infty$, $\boldsymbol{V}_0$ are well-defined and bounded. Moreover, the space $H^2(\mathbb{R}^N)$ is continuously embedded into $\mathcal{D}(\boldsymbol{V}_0)$, and the linear operator $\boldsymbol{V} = \boldsymbol{V}_\infty + \boldsymbol{V}_0$ is well-defined. In addition, operator $\boldsymbol{V}$ considered as a linear operator from the space $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ is bounded, and, for any $u \in H^2(\mathbb{R}^N)$,

$$\|\boldsymbol{V} u\|_{L^2} \leq \|V_\infty\|_{L^\infty} \, \|u\|_{L^2} + C\|V_0\|_{L^p} \, \|u\|_{H^2}$$

where $C = C(N, p) > 0$ is a constant.

(ii) The restricted linear operator $\boldsymbol{V}_0|_{H^2(\mathbb{R}^N)}$ is compact, that is, the set $\boldsymbol{V}_0(U)$ is relatively compact in $L^2(\mathbb{R}^N)$, for any bounded $U \subset H^2(\mathbb{R}^N)$.

We are in a position to prove the first main result of this section, which collects the properties of $\boldsymbol{V}_0$, A_∞ and A.

Proposition 2.1.10.

(i) $\boldsymbol{V}_0$ is relatively $(-\Delta)$-compact, i.e., $\boldsymbol{V}_0 : (H^2(\mathbb{R}^N), \|\cdot\|_{-\Delta}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{L^2})$ is a compact linear operator (see Appendix A.1).

(ii) $A = -\Delta + \boldsymbol{V}$ with $D(A) = H^2(\mathbb{R}^N)$ and $A_\infty = -\Delta + \boldsymbol{V}_\infty$ with $D(A_\infty) = H^2(\mathbb{R}^N)$ are well-defined linear operators in $L^2(\mathbb{R}^N)$. Furthermore, $\boldsymbol{V}_0$ is relatively A_∞-compact and A, A_∞ are closed and self-adjoint operators in $L^2(\mathbb{R}^N)$.

The proof makes use of the following remark.

Remark 2.1.11.

(i) Let $\|\cdot\|_{-\Delta}$ be the graph norm on $D(-\Delta) = H^2(\mathbb{R}^N)$:

$$\|u\|_{-\Delta} = \left(\|u\|_{L^2}^2 + \|(-\Delta)u\|_{L^2}^2\right)^{1/2}.$$

We claim that $\|\cdot\|_{-\Delta}$ and $\|\cdot\|_{H^2}$ are equivalent. Indeed, for $u \in H^2(\mathbb{R}^N)$, one has

$$\|u\|_{-\Delta} \le \|(-\Delta)u\|_{L^2} + \|u\|_{L^2} \le \|u\|_{H^2}.$$

Hence, the identity mapping

$$j : \left(H^2(\mathbb{R}^N), \|\cdot\|_{H^2}\right) \to \left(H^2(\mathbb{R}^N), \|\cdot\|_{-\Delta}\right)$$

is continuous. The operator $-\Delta$ is closed, so the space $\left(H^2(\mathbb{R}^N), \|\cdot\|_{-\Delta}\right)$ is complete, and, in view of the Banach Isomorphism Theorem (see [9, Corollary 2.7]), j^{-1} is bounded, i.e., the norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{-\Delta}$ are equivalent.

(ii) Let $\|\cdot\|_{-\Delta+\boldsymbol{V}_\infty}$ be the graph norm on $D(-\Delta + \boldsymbol{V}_\infty) = H^2(\mathbb{R}^N)$:

$$\|u\|_{-\Delta+\boldsymbol{V}_\infty} = \left(\|u\|_{L^2}^2 + \|(-\Delta + \boldsymbol{V}_\infty)u\|_{L^2}^2\right)^{1/2}.$$

Then, $\|\cdot\|_{-\Delta+\boldsymbol{V}_\infty}$ and $\|\cdot\|_{-\Delta}$ are equivalent. In fact, observe that

$$\begin{aligned}\|u\|_{-\Delta} &\le \|u\|_{L^2} + \|-\Delta u\|_{L^2} \le \|u\|_{L^2} + \|-\boldsymbol{V}_\infty u\|_{L^2} + \|(-\Delta + \boldsymbol{V}_\infty)u\|_{L^2} \\ &\le \left(1 + \|V_\infty\|_{L^\infty}\right)\Big(\|u\|_{L^2} + \|(-\Delta + \boldsymbol{V}_\infty)u\|_{L^2}\Big) \\ &= \sqrt{2}\,\left(1 + \|V_\infty\|_{L^\infty}\right)\,\|u\|_{-\Delta+\boldsymbol{V}_\infty}.\end{aligned} \tag{2.19}$$

On the other hand, in a similar manner we get

$$\begin{aligned}\|u\|_{-\Delta+\boldsymbol{V}_\infty} &\le \|u\|_{L^2} + \|-\Delta u + \boldsymbol{V}_\infty u\|_{L^2} \le (1 + \|V_\infty\|_{L^\infty})\big(\|u\|_{L^2} + \|-\Delta u\|_{L^2}\big) \\ &\le \sqrt{2}\,(1 + \|V_\infty\|_{L^\infty})\,\|u\|_{-\Delta}.\end{aligned} \tag{2.20}$$

□

Proof of Proposition 2.1.10. (i) We shall prove that the linear operator

$$\widetilde{\boldsymbol{V}}_0 : (H^2(\mathbb{R}^N), \|\cdot\|_{-\Delta}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{L^2}),$$

given by $\widetilde{\boldsymbol{V}}_0 u = \boldsymbol{V}_0 u$ for $u \in H^2(\mathbb{R}^N)$, is compact. By Remark 2.1.11 (i), the identity map

$$j : (H^2(\mathbb{R}^N), \|\cdot\|_{-\Delta}) \to (H^2(\mathbb{R}^N), \|\cdot\|_{H^2})$$

is a bounded linear operator. Moreover, by Corollary 2.1.9 (ii), the linear operator $\boldsymbol{V}_0|_{H^2(\mathbb{R}^N)}$ is compact. Since $\widetilde{\boldsymbol{V}}_0 = \boldsymbol{V}_0|_{H^2(\mathbb{R}^N)} \circ j$, we get that $\widetilde{\boldsymbol{V}}_0$ is a compact linear operator.

(ii) In view of Corollary 2.1.9 (i), $\boldsymbol{V}_\infty$, $\boldsymbol{V}_0$ and $\boldsymbol{V}$ are well-defined on $H^2(\mathbb{R}^N)$, thus, so are $-\Delta + \boldsymbol{V}_\infty$, $-\Delta + \boldsymbol{V}_0$ and $-\Delta + \boldsymbol{V}$. By the point (i), $\boldsymbol{V}_0$ is relatively $-\Delta$-compact. On the other hand, from Remark 2.1.11 (ii) we deduce that the norms $\|\cdot\|_{-\Delta+\boldsymbol{V}_\infty}$ and $\|\cdot\|_{-\Delta}$ are equivalent. Therefore, $\boldsymbol{V}_0$ is relatively $(-\Delta + \boldsymbol{V}_\infty)$-compact. The operator $-\Delta$ is self-adjoint and $\boldsymbol{V}_\infty$ is a bounded symmetric operator, hence, $A_\infty = -\Delta + \boldsymbol{V}_\infty$ is self-adjoint (see Theorem A.1.5 (ii)). The operator $\boldsymbol{V}_0$ is relatively A_∞-compact, thus, $\boldsymbol{V}_0$ is relatively A_∞-bounded and its A_∞-bound equals zero (see Theorem A.1.4). Consequently, because A_∞ is self-adjoint and $\boldsymbol{V}_0$ is symmetric, based on the Kato-Rellich Theorem (Theorem A.1.5 (ii)), we infer that $A = A_\infty + \boldsymbol{V}_0$ is self-adjoint. □

Now, we state the second main result of this section.

Proposition 2.1.12. *For the operator $A = -\Delta + \boldsymbol{V}$ there holds*

(i) $\sigma_{\mathrm{ess}}(A) \subset [\varrho(V_\infty), +\infty)$, where $\sigma_{\mathrm{ess}}(A)$ denotes the essential spectrum of the operator A (cf. Appendix A.1) and $\varrho(V_\infty)$ denotes the asymptotic bottom (see (6)*) of the $L^\infty(\mathbb{R}^3)$-part of the potential V, denoted by V_∞, that is*

$$\varrho(V_\infty) = \lim_{R\to+\infty} \operatorname*{ess\,inf}_{\mathbb{R}^N\setminus D(0,R)} V_\infty;$$

(ii) there exists a number $c \in \mathbb{R}$ such that $\sigma(A) \subset [c, +\infty)$;

(iii) for any $\eta < \varrho(V_\infty)$ the set $\sigma(A) \cap (-\infty, \eta]$ consists of a finite number of eigenvalues with finite multiplicities(2), i.e., $\sigma(A) \cap (-\infty, \eta] \subset \sigma_{\mathrm{disc}}(A)$, where $\sigma_{\mathrm{disc}}(A)$ denotes the discrete spectrum of the operator A (see Appendix A.1).

Before the proof we need two lemmata.

Lemma 2.1.13. *Assume that $V_0 \in L^p(\mathbb{R}^N)$, with $p \geq 2$ as in* (5b)*.*

(i) The space $H^1(\mathbb{R}^N)$ is continuously embedded into $L^{2p/(p-1)}(\mathbb{R}^N)$ and, for $u \in H^1(\mathbb{R}^N)$,

$$\|V_0\,|u|^2\|_{L^1} \leq C^2\ \|V_0\|_{L^p}\ \|u\|^2_{H^1}$$

where $C = C(p, N) > 0$ is a constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2p/(p-1)}(\mathbb{R}^N)$.

(ii) For any $u \in H^1(\mathbb{R}^N)$

$$\|V_0\,|u|^2\|_{L^1} \leq C^{2\theta}\ \|V_0\|_{L^p}\ \|u\|^{2\theta}_{H^1}\ \|u\|^{2(1-\theta)}_{L^2}. \tag{2.21}$$

Here $C = C(p, r, N) > 0$ is a constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, where

$$\begin{cases} r > 2p/(p-1), & \theta = r/p(r-2), \quad \theta \in (0,1) \quad \textit{if} \quad N \in \{1,2\} \\ r = 2N/(N-2), & \theta = (N/2)/p, \quad \theta \in (0,1] \quad \textit{if} \quad N \geq 3. \end{cases}$$

(2)Since operator A is self-adjoint, for any eigenvalue λ of A the geometric multiplicity of λ equals its algebraic multiplicity (cf. Appendix A.1).

(iii) If $N \geq 5$ and $p = N/2$, and $U \subset \mathbb{R}^N$ has finite measure, then, for any $u \in H^1(\mathbb{R}^N)$ and $b \in (0,1)$,

$$\|V_0\,|u|^2\|_{L^1(U)} \leq C^2\,\||V_0| - |V_0|^b\|_{L^{N/2}(U)}\,\|u\|_{H^1}^2 + C_1\,\|u\|_{H^1}^{2b}\,\|u\|_{L^2}^{2(1-b)} \tag{2.22}$$

where $C = C(N) > 0$ is a constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2N/(N-2)}(\mathbb{R}^N)$ and $C_1 = \max\{1, C^2\} \cdot \max\{1, \|V_0\|_{L^{N/2}}\}$.

Proof. (i) Let $p \geq 2$ be as in (5b). Then $p \geq 2$ if $N \in \{1,2\}$ and $p \geq N/2$ if $N \geq 3$. Hence, Remark 2.1.2 yields the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2p/(p-1)}(\mathbb{R}^N)$. Consequently, using the Hölder inequality,

$$\|V_0\,|u|^2\|_{L^2} \leq \|V_0\|_{L^p}\,\|u\|_{L^{2p/(p-1)}}^2, \tag{2.23}$$

we prove the required inequality.

(ii) We take $r \geq 2p/(p-1)$ and, in view of the interpolation inequality (cf. Proposition A.2.2 (i)), we obtain

$$\|u\|_{L^{2p/(p-1)}}^2 \leq \|u\|_{L^2}^{2(1-\theta)}\,\|u\|_{L^r}^{2\theta} \tag{2.24}$$

with $\theta = r/p(r-2)$. Note that $\theta \in (0,1]$.

Let $N \in \{1,2\}$ and fix $r > 2p/(p-1)$. Based on Remark 2.1.2, one has the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$. Then, combining (2.23) with (2.24), we get the desired inequality. In addition, $\theta \in (0,1)$, because $r > 2p/(p-1)$.

Assume that $N \geq 3$ and let $r = 2N/(N-2)$. In the light of Remark 2.1.2, there is the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$. Further, by virtue of (5b), $p \geq N/2$, hence, $r \geq 2p/(p-1)$ and from (2.23) and (2.24) we deduce (2.21).

(iii) By assumption, the measure of U is finite and $(N/2)/b > N/2$, thus $|V_0|^b\,\mathbf{1}_U \in L^{N/2}(\mathbb{R}^N)$. By (i), $|V_0|^b\,\mathbf{1}_U\,|u|^2 \in L^1(\mathbb{R}^N)$, therefore, we obtain

$$\|V_0\,|u|^2\|_{L^1(U)} \leq \left\||V_0|\,\mathbf{1}_U\,|u|^2 - |V_0|^b\,\mathbf{1}_U\,|u|^2\right\|_{L^1} + \||V_0|^b\,\mathbf{1}_U\,|u|^2\|_{L^1}. \tag{2.25}$$

Applying (ii) twice, firstly for $p = N/2$, we get

$$\left\|\left(|V_0|\,\mathbf{1}_U - |V_0|^b\,\mathbf{1}_U\right)|u|^2\right\|_{L^1} \leq C^2\,\left\||V_0| - |V_0|^b\right\|_{L^{N/2}(U)}\,\|u\|_{H^1}^2,$$

and secondly, for $p = (N/2)/b$, we get

$$\begin{aligned}\||V_0|^b\,\mathbf{1}_U\,|u|^2\|_{L^1} &\leq C^{2b}\,\||V_0|^b\,\mathbf{1}_U\|_{L^{(N/2)/b}}\,\|u\|_{H^1}^{2b}\,\|u\|_{L^2}^{2(1-b)}\\ &\leq \max\{1, C^2\}\,\max\{1, \|V_0\|_{L^{N/2}}\}\,\|u\|_{H^1}^{2b}\,\|u\|_{L^2}^{2(1-b)}\end{aligned}$$

where $C > 0$ is the constant from the continuous embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2N/(N-2)}(\mathbb{R}^N)$. Combining these inequalities with (2.25), we obtain (2.22). □

Lemma 2.1.14. (compare with [48, inequality (1.2) on p. 145]) *Assume that $V_0 \in L^p(\mathbb{R}^N)$ with $p \geq 2$ as in* (5b). *For any $\varepsilon > 0$ there exists a constant $C = C(p, N, \varepsilon) > 0$ such that*

$$\left|\int_{\mathbb{R}^N} V_0(x)|u(x)|^2\,dx\right| \leq \varepsilon\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^2 \quad \textit{for all } u \in H^1(\mathbb{R}^N).$$

In particular, for all $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2\,dx \geq (1/2)\|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^2.$$

Proof. By Lemma 2.1.13 (ii), there exist $C > 0$ and $\theta \in (0,1]$ such that, for all $u \in H^1(\mathbb{R}^N)$,

$$\|V_0|u|^2\|_{L^1} \leq C^{2\theta}\,\|V_0\|_{L^p}\,\|u\|_{H^1}^{2\theta}\,\|u\|_{L^2}^{2(1-\theta)}.$$

Suppose that $N \in \{1,2,3,4\}$ or $N \geq 5$ and $p > N/2$. Let $\varepsilon > 0$ be arbitrary. We see that $\theta \in (0,1)$ for $N \in \{1,2\}$ and $N \geq 5$, $p > N/2$. Further, $\theta \in (0,1)$ also for $N = 3$, because then $p \geq 2 > N/2$, and for $N = 4$, because then $p > 2 = N/2$. Consequently, applying Young's inequality (Lemma A.2.1 (i)), we get

$$\begin{aligned}\|V_0\,|u|^2\|_{L^1} &\leq C^{2\theta}\,\|V_0\|_{L^p}\left(\frac{\varepsilon\|u\|_{H^1}^2}{1/\theta} + \frac{\|u\|_{L^2}^2}{\varepsilon^{1/(1-\theta)-1}\,1/(1-\theta)}\right)\\ &\leq \varepsilon\,C^{2\theta}\,\|V_0\|_{L^p}\,\|u\|_{H^1}^2 + C^{2\theta}\,\|V_0\|_{L^p}\,\varepsilon^{1-1/(1-\theta)}\,\|u\|_{L^2}^2,\end{aligned}$$

which implies the assertion, because C and θ do not depend on ε.

Assume now that $N \geq 5$ and $p = N/2$, and let $\varepsilon > 0$. There exists $r > 0$ such that

$$\|V_0\|_{L^{N/2}(\mathbb{R}^N\setminus B(0,r))} \leq \varepsilon.$$

Then we apply the Hölder inequality to obtain

$$\|V_0\,|u|^2\|_{L^1(\mathbb{R}^N\setminus B(0,r))} \leq \|V_0\|_{L^{N/2}(\mathbb{R}^N\setminus B(0,r))}\,\|u\|_{L^{2N/(N-2)}(\mathbb{R}^N\setminus B(0,r))}^2 \leq \varepsilon\,C^2\,\|u\|_{H^1}^2$$

where $C = C(N) > 0$ is a constant from the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2N/(N-2)}(\mathbb{R}^N)$. This yields

$$\|V_0\,|u|^2\|_{L^1} \leq \|V_0\,|u|^2\|_{L^1(B(0,r))} + \|V_0\,|u|^2\|_{L^1(\mathbb{R}^N\setminus B(0,r))} \leq \|V_0\,|u|^2\|_{L^1(B(0,r))} + \varepsilon\,C^2\,\|u\|_{H^1}^2. \tag{2.26}$$

In view of Lemma 2.1.13 (iii), there exists a constant $C_1 > 0$ such that, for any $b \in (0,1)$ and any $u \in H^1(\mathbb{R}^N)$, we obtain

$$\|V_0\,|u|^2\|_{L^1(B(0,r))} \leq C^2\,\||V_0| - |V_0|^b\|_{L^{N/2}(B(0,r))}\,\|u\|_{H^1}^2 + C_1\,\|u\|_{H^1}^{2b}\,\|u\|_{L^2}^{2(1-b)}. \tag{2.27}$$

By Lemma 2.1.8, we set $b \in (0,1)$ such that

$$\||V_0| - |V_0|^b\|_{L^{N/2}(B(0,r))} \leq \varepsilon. \tag{2.28}$$

From Young's inequality we deduce that

$$\|u\|_{H^1}^{2b}\,\|u\|_{L^2}^{2(1-b)} \leq \frac{\varepsilon\|u\|_{H^1}^2}{1/b} + \frac{\|u\|_{L^2}^2}{\varepsilon^{1/(1-b)-1}\,1/(1-b)} \leq \varepsilon\|u\|_{H^1}^2 + \varepsilon^{1-1/(1-b)}\,\|u\|_{L^2}^2.$$

It follows from (2.27) and (2.28) that

$$\|V_0\,|u|^2\|_{L^1(B(0,r))} \leq \varepsilon C^2\,\|u\|_{H^1}^2 + C_1\varepsilon\|u\|_{H^1}^2 + C_1\,\varepsilon^{1-1/(1-b)}\,\|u\|_{L^2}^2,$$

which together with (2.26) proves the assertion, because C and C_1 do not depend on ε. □

Proof of Proposition 2.1.12. (i) In view of the Persson formula (see [48, Thm. 2.1], [36, Thm. 14.11]) one has

$$\inf\,\sigma_{\text{ess}}\,(A_\infty) = \lim_{R\to+\infty}\,\inf\left\{\langle A_\infty u, u\rangle_{L^2} : u \in C_0^\infty\big(\mathbb{R}^N \setminus D(0,R)\big), \|u\|_{L^2} = 1\right\} \tag{2.29}$$

where $A_\infty = -\Delta + \boldsymbol{V}_\infty$ (see (2.7)). Let $u \in C_0^\infty\big(\mathbb{R}^N \setminus D(0,R)\big)$ be such that $\|u\|_{L^2} = 1$. Then

$$\langle(-\Delta + \boldsymbol{V}_\infty)u, u\rangle_{L^2} = \|\nabla u\|_{L^2}^2 + \langle \boldsymbol{V}_\infty u, u\rangle_{L^2} \geq \int_{\mathbb{R}^N\setminus D(0,R)} V_\infty(x)|u(x)|^2\,dx. \tag{2.30}$$

For almost every $x \in \mathbb{R}^N \setminus D(0,R)$ we have

$$V_\infty(x)|u(x)|^2 \geq \operatorname*{ess\,inf}_{\mathbb{R}^N\setminus D(0,R)} V_\infty\;\;|u(x)|^2,$$

thus, since $\|u\|_{L^2} = 1$,

$$\int_{\mathbb{R}^N \setminus D(0,R)} V_\infty(x)|u(x)|^2\, dx \geq \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty.$$

Taking into account (2.30), we obtain

$$\langle(-\Delta + \boldsymbol{V}_\infty)u, u\rangle_{L^2} \geq \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty$$

and consequently

$$\inf\Big\{\langle(-\Delta + \boldsymbol{V}_\infty)u, u\rangle_{L^2} : u \in C_0^\infty\big(\mathbb{R}^N \setminus D(0,R)\big), \|u\|_{L^2} = 1\Big\} \geq \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty.$$

Combining this with (2.29) and the definition of asymptotic bottom, we arrive at

$$\inf \sigma_{\mathrm{ess}}(-\Delta + \boldsymbol{V}_\infty) \geq \varrho(V_\infty),$$

therefore

$$\sigma_{\mathrm{ess}}(-\Delta + \boldsymbol{V}_\infty) \subset [\varrho(V_\infty), +\infty). \tag{2.31}$$

Next, in view of Proposition 2.1.10 (ii), the symmetric operator $\boldsymbol{V}_0$ is relatively $(-\Delta + \boldsymbol{V}_\infty)$-compact, hence, the Weyl's Theorem (Theorem A.1.5 (iii)) yields

$$\sigma_{\mathrm{ess}}(-\Delta + \boldsymbol{V}) = \sigma_{\mathrm{ess}}(-\Delta + \boldsymbol{V}_\infty).$$

Taking into considerations (2.31) we get the assertion (i).

(ii) By Proposition 2.1.10 (ii), the operator $-\Delta + \boldsymbol{V}$ is self-adjoint, thus, in view of the spectral bottom formula (see Theorem A.1.5 (i)), it suffices to prove that there exists $c \in \mathbb{R}$ such that, $\langle(-\Delta+\boldsymbol{V})u+cu, u\rangle_{L^2} \geq 0$ for any $u \in H^2(\mathbb{R}^N)$. Note that, by virtue of Lemma 2.1.14, there exists $C > 0$ such that, for all $u \in H^2(\mathbb{R}^N)$,

$$\langle(-\Delta + \boldsymbol{V})u + Cu, u\rangle_{L^2} = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2\, dx + C\|u\|_{L^2}^2 \geq (1/2)\|\nabla u\|_{L^2}^2 \geq 0.$$

Therefore, one has $\sigma(-\Delta + \boldsymbol{V}) \subset [-C, +\infty)$.

(iii) We set some $\eta < \varrho(V_\infty)$. Then $\sigma(A) \cap (-\infty, \eta]$ is bounded, because, by (ii), $\sigma(A) \subset [c, +\infty)$ for some $c \in \mathbb{R}$. Moreover, $\sigma_{\mathrm{ess}}(A) \subset [\varrho(V_\infty), +\infty)$, thus $\sigma(A) \cap (-\infty, \eta]$ consists of isolated eigenvalues with finite multiplicities. We claim that $\sigma(A) \cap (-\infty, \eta]$ has no accumulation points. Indeed, suppose to the contrary that $x_0 \in \mathbb{R}$ is a such point. Since $\sigma(A) \cap (-\infty, \eta]$ is closed, we get $x_0 \in \sigma(A) \cap (-\infty, \eta]$. Hence, x_0 is not an isolated point of the spectrum $\sigma(A)$ and $x_0 < \varrho(V_\infty)$, which contradicts the property that $\sigma_{\mathrm{ess}}(A) \subset [\varrho(V_\infty), +\infty)$. Therefore, $\sigma(A) \cap (-\infty, \eta]$ is bounded and has no accumulation points, so it must be finite. □

Example 2.1.15. Consider the linear operator $B : H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ given by

$$B = -p^2\Delta - \frac{q^2}{|x|}$$

where p, $q > 0$ are constants. It is known (cf. [34, Thm. 18.3 and Thm. 18.4]) that all negative eigenvalues of the operator B are of the form

$$\lambda_n = -\frac{q^4}{4p^2n^2}, \quad n \geq 1.$$

Using [36, Corollary 14.10], we get

$$\sigma_{\mathrm{disc}}(B) = \{\lambda_n : n \geq 1\} \ \text{ and } \ \sigma_{\mathrm{ess}}(B) = [0, +\infty). \tag{2.32}$$

Additionally (cf. [34, Corollary 18.5]), for any $n \geq 1$, one has

$$\dim \operatorname{Ker}(\lambda_n I - B) = n^2. \tag{2.33}$$

Now, consider the Schrödinger operator $-\Delta + \boldsymbol{V}$, where $\boldsymbol{V} : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the operator of multiplication by the function $V(x) = -a/|x| + b$, where $a = q^2/p$ and $b \in \mathbb{R}$ is a constant. We will show that

$$\sigma_{\text{disc}}(-\Delta + \boldsymbol{V}) = \left\{ -\frac{a^2}{4n^2} + b : n \geq 1 \right\} \quad \text{and} \quad \sigma_{\text{ess}}(-\Delta + \boldsymbol{V}) = [b, +\infty) \tag{2.34}$$

and

$$\dim \operatorname{Ker}\Big(\lambda_n I - (-\Delta + \boldsymbol{V})\Big) = n^2 \quad \text{for } n \geq 1. \tag{2.35}$$

To this end, we define the Schrödinger operator $-\Delta + \boldsymbol{W}$, where $\boldsymbol{W} : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the operator of multiplication by the function $W(x) = -a/|x|$. By virtue of Example 2.1.1, V and W are Kato-Rellich type potentials, hence, in view of Proposition 2.1.10 (ii), the linear operators $-\Delta + \boldsymbol{V}$ and $-\Delta + \boldsymbol{W}$ are well-defined. We claim that

$$\sigma_{\text{disc}}(-\Delta + \boldsymbol{W}) = \sigma_{\text{disc}}(B) \quad \text{and} \quad \sigma_{\text{ess}}(-\Delta + \boldsymbol{W}) = \sigma_{\text{ess}}(B) \tag{2.36}$$

and

$$\dim \operatorname{Ker}\Big(\lambda_n I - (-\Delta + \boldsymbol{W})\Big) = \dim \operatorname{Ker}(\lambda_n I - B) \quad \text{for } n \geq 1. \tag{2.37}$$

Given $s > 0$, let us define the linear operator $R_s : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by the formula

$$[R_s u](x) = u(sx).$$

The change of variables formula yields

$$\|R_s u\|_{L^2} = \frac{1}{s\sqrt{s}} \|u\|_{L^2} \quad \text{for } u \in L^2(\mathbb{R}^3).$$

Hence, the operator R_s is well-defined and bounded. Moreover, by directly using the definition of the weak derivative and changing variables in the integrals, we obtain that

$$\frac{\partial}{\partial x_i}(R_s u) = -s R_s \frac{\partial}{\partial x_i} u \quad \text{for } i \in \{1,2,3\} \text{ and } u \in H^1(\mathbb{R}^3)$$

and

$$\frac{\partial^2}{\partial x_j \partial x_i}(R_s u) = s^2 R_s \frac{\partial^2}{\partial x_j \partial x_i} u \quad \text{for } i, j \in \{1,2,3\} \text{ and } u \in H^2(\mathbb{R}^3).$$

In particular, this means that $R_s(H^2(\mathbb{R}^3)) \subset H^2(\mathbb{R}^3)$ and

$$\Delta R_s u = s^2 R_s(\Delta u) \quad \text{for } u \in H^2(\mathbb{R}^3). \tag{2.38}$$

Observe that, for any $s > 0$, the operator R_s is invertible and $R_s^{-1} = R_{1/s}$. This shows immediately that R_s^{-1} is also a bounded linear operator.

We claim that

$$-\Delta + \boldsymbol{W} = R_p B R_p^{-1}. \tag{2.39}$$

Indeed, defining the mapping $U(x) = -q^2/|x|$ and $\boldsymbol{U} : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as the operator of multiplication by the function U, one has, for any $u \in H^2(\mathbb{R}^3)$ and almost every $x \in \mathbb{R}^3$,

$$\Big[R_p \boldsymbol{U} R_p^{-1}\, u\Big](x) = \Big[\boldsymbol{U} R_p^{-1}\, u\Big](px) = U(px)[R_p^{-1}\, u](px) = W(x)u(x) = [\boldsymbol{W}u](x). \tag{2.40}$$

Applying (2.38) together with (2.40), we arrive at

$$\begin{aligned} R_p B R_p^{-1} &= R_p\left(-p^2\Delta + \boldsymbol{U}\right) R_p^{-1} = \left(-p^2 R_p \Delta + R_p \boldsymbol{U}\right) R_p^{-1} \\ &\overset{(2.38)}{=} \left(-\Delta R_p + R_p \boldsymbol{U}\right) R_p^{-1} \overset{(2.40)}{=} -\Delta + \boldsymbol{W} \end{aligned}$$

as desired.

Having the formula (2.39), we apply Proposition A.1.7 to get (2.36) and (2.37). We easily see that (2.32) and (2.36) together imply (2.34), while (2.33) and (2.37) imply (2.35). ☐

2.2 Damped wave operator

We consider the space $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with the scalar product:

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathbb{X}} = \langle u_1, u_2 \rangle_{H^1} + \langle v_1, v_2 \rangle_{L^2} \quad \text{for} \quad (u_1, v_1), (u_2, v_2) \in \mathbb{X},$$

which induces the norm

$$\|(u, v)\|_{\mathbb{X}}^2 = \|u\|_{H^1}^2 + \|v\|_{L^2}^2 \quad \text{for} \quad (u, v) \in \mathbb{X}.$$

Since $H^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ are Hilbert spaces, then $\mathbb{X}$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ is Hilbert space. Now, we define the *damped wave operator* $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ by the formula

$$\mathbb{A}(u, v) = (v, -Au - \beta v) \quad \text{for} \quad (u, v) \in D(\mathbb{A}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N). \tag{2.41}$$

In view of Proposition 2.1.10 (ii), the operator $A : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $A = -\Delta + \boldsymbol{V}$, is well-defined, therefore $\mathbb{A}$ is so.

Proposition 2.2.1.

(i) $\mathbb{A}$ *is densely defined and closed operator.*

(ii) $\mathbb{A}$ *generates a* C_0*-semigroup* $\{e^{t\mathbb{A}}\}_{t \geq 0}$ *on* $\mathbb{X}$.

(iii) $\sigma(\mathbb{A}) = s_\beta^{-1}(\sigma(A))$, *where* $s_\beta : \mathbb{C} \to \mathbb{C}$ *is given by* $s_\beta(\mu) = -\mu(\mu + \beta)$ *and* $s_\beta^{-1}(\{\lambda\}) = \{\mu_-(\lambda), \mu_+(\lambda)\}$, *where*

$$\mu_\pm(\lambda) = \begin{cases} -\beta/2 \pm \sqrt{(\beta/2)^2 - \lambda} & \text{if } \lambda \leq (\beta/2)^2, \\ -\beta/2 \pm i\sqrt{\lambda - (\beta/2)^2} & \text{if } \lambda > (\beta/2)^2. \end{cases} \tag{2.42}$$

(iv) For any $\mu \in \mathbb{C}$ *there holds*

$$\operatorname{Ker}(\mu I - \mathbb{A}) = \{(u, \mu u) \,:\, u \in \operatorname{Ker}(s_\beta(\mu) I - A)\}. \tag{2.43}$$

Moreover, we have $\sigma_p(\mathbb{A}) = s_\beta^{-1}(\sigma_p(A))$, *where* σ_p *refers to the point spectrum of linear operator (cf. Appendix A.1).*

Proof. (i) $\mathbb{A}$ is densely defined, because, due to Friedrich's Theorem (see Theorem A.2.3), $H^2(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ and, due to Proposition A.2.2 (iii), $H^1(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$.

Let $\big((u_n, v_n)\big)_{n \geq 1}$ be a sequence in $D(\mathbb{A})$ such that

$$(u_n, v_n) \xrightarrow{n \to \infty} (u, v) \quad \text{in } \mathbb{X}. \tag{2.44}$$

and

$$\mathbb{A}(u_n, v_n) \xrightarrow{n \to \infty} (g, h) \quad \text{in } \mathbb{X}. \tag{2.45}$$

From (2.44) we deduce that $u_n \xrightarrow{n \to \infty} u$ in $H^1(\mathbb{R}^N)$ and $v_n \xrightarrow{n \to \infty} v$ in $L^2(\mathbb{R}^N)$, and (2.45) yields $v_n \xrightarrow{n \to \infty} g$ in $H^1(\mathbb{R}^N)$ and $-Au_n - \beta v_n \xrightarrow{n \to \infty} h$ in $L^2(\mathbb{R}^N)$. This implies that $v = g$ and $Au_n \xrightarrow{n \to \infty} -\beta v - h$ in $L^2(\mathbb{R}^N)$. By Proposition 2.1.10 (ii), A is closed, therefore, we get $u \in H^2(\mathbb{R}^N)$ and $Au = -\beta v - h$, i.e., $(u, v) \in D(\mathbb{A})$ and $\mathbb{A}(u, v) = (v, -Au - \beta v) = (g, h)$.

(ii) We define the operator $\mathbb{A}_0 : D(\mathbb{A}_0) \subset \mathbb{X} \to \mathbb{X}$,

$$\mathbb{A}_0(u, v) = (v, \Delta u - u - \beta v) \quad \text{for} \quad (u, v) \in D(\mathbb{A}_0) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N).$$

We claim that $\mathbb{A}_0$ is a maximal dissipative operator. Indeed, firstly, for $(u, v) \in D(\mathbb{A}_0)$, one has

$$\begin{aligned} \langle \mathbb{A}_0(u, v), (u, v) \rangle_{\mathbb{X}} &= \big\langle (v, \Delta u - u - \beta v), (u, v) \big\rangle_{\mathbb{X}} \\ &= \langle v, u \rangle_{H^1} + \langle \Delta u, v \rangle_{L^2} - \langle u, v \rangle_{L^2} - \beta \|v\|_{L^2}^2 = -\beta \|v\|_{L^2}^2 \leq 0. \end{aligned}$$

Secondly, let $(g,h)\in\mathbb{X}$ and we put

$$u=\big(\Delta-(2+\beta)I\big)^{-1}\big((1+\beta)g+h\big) \quad\text{and}\quad v=g+u.$$

By [36, Thm. 7.6], $(0,+\infty)\subset\rho(\Delta)$, thus, the vectors u, v are well-defined and $(u,v)\in D(\mathbb{A}_0)$. Furthermore, one readily gets

$$(\mathbb{A}_0-I)(u,v)=(v-u,\Delta u-u-\beta v-v)=\big(g,(\Delta-(2+\beta)I)u-(1+\beta)g\big)=(g,h).$$

Hence, $\operatorname{Im}(\mathbb{A}_0-I)=\mathbb{X}$. This means that $\mathbb{A}_0$ is a maximal dissipative operator and, by the Lumer-Phillips Theorem (Theorem 1.1.3), $\mathbb{A}_0$ is the generator of a C_0-semigroup of contractions.

Next, let $\mathbb{V}:D(\mathbb{V})\subset\mathbb{X}\to\mathbb{X}$ be a linear operator given by

$$\mathbb{V}(u,v)=(0,u-\boldsymbol{V}u) \quad\text{for}\quad (u,v)\in D(\mathbb{V})=H^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N).$$

We shall show that $\mathbb{V}$ is $\mathbb{A}_0$-bounded. In view of Lemma 2.1.9, the linear operator $\boldsymbol{V}$ is bounded as an operator from $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$, hence, for all $(u,v)\in H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$,

$$\begin{aligned}\|\mathbb{V}(u,v)\|_{\mathbb{X}}&=\|u-\boldsymbol{V}u\|_{L^2}\le\|u\|_{L^2}+\|\boldsymbol{V}u\|_{L^2}\\&\le(1+\|\boldsymbol{V}\|_{\mathcal{L}(H^2,L^2)})\,\|u\|_{H^2}\le(1+\|\boldsymbol{V}\|_{\mathcal{L}(H^2,L^2)})\,\|(u,v)\|_{H^2\times H^1}\end{aligned}\tag{2.46}$$

where $\|\cdot\|_{H^2\times H^1}$ is the norm on $H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ generated by the scalar product

$$\langle(u_1,v_1),(u_2,v_2)\rangle_{H^2\times H^1}=\langle u_1,u_2\rangle_{H^2}+\langle v_1,v_2\rangle_{H^1}\quad\text{for } (u_1,v_1),(u_2,v_2)\in H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N).$$

We assert that the norms $\|\cdot\|_{H^2\times H^1}$ and $\|\cdot\|_{\mathbb{A}_0}$ are equivalent, where $\|\cdot\|_{\mathbb{A}_0}$ is the graph norm on $D(\mathbb{A}_0)$. Indeed, for $(u,v)\in H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, one has

$$\begin{aligned}\|\mathbb{A}_0(u,v)\|_{\mathbb{X}}&\le\|v\|_{H^1}+\|\Delta u-u-\beta v\|_{L^2}\le\|v\|_{H^1}+\beta\|v\|_{L^2}+\|\Delta u\|_{L^2}+\|u\|_{L^2}\\&\le(1+\beta)\|v\|_{H^1}+\sqrt{2}\left(\|\Delta u\|_{L^2}^2+\|u\|_{L^2}^2\right)^{1/2}\\&\le(1+\beta)\|v\|_{H^1}+\sqrt{2}\|u\|_{H^2}\le(\sqrt{2}+\beta)\|(u,v)\|_{H^2\times H^1},\end{aligned}$$

which implies

$$\begin{aligned}\|(u,v)\|_{\mathbb{A}_0}&=\left(\|\mathbb{A}_0(u,v)\|_{\mathbb{X}}^2+\|(u,v)\|_{\mathbb{X}}^2\right)^{1/2}\\&\le\left(\left(\sqrt{2}+\beta\right)^2\|(u,v)\|_{H^2\times H^1}^2+\|(u,v)\|_{H^2\times H^1}^2\right)^{1/2}\le(\sqrt{2}+\beta)\|(u,v)\|_{H^2\times H^1}.\end{aligned}$$

From this, we deduce that the identity map

$$j:\big(H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N),\|\cdot\|_{H^2\times H^1}\big)\to\big(H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\big)$$

is continuous. Since $(H^2(\mathbb{R}^N),\langle\cdot,\cdot\rangle_{H^2})$ and $(H^1(\mathbb{R}^N),\langle\cdot,\cdot\rangle_{H^1})$ are Hilbert spaces, then so is $\big(H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N),\langle\cdot,\cdot\rangle_{H^2\times H^1}\big)$. As for the operator $\mathbb{A}$, we can show that $\mathbb{A}_0$ is a closed linear operator, therefore, the space $\big(H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N),\|\cdot\|_{\mathbb{A}_0}\big)$ is complete. Then, by virtue of the Banach Isomorphism Theorem (see [9, Corollary 2.7]), the norms $\|\cdot\|_{H^2\times H^1}$ and $\|\cdot\|_{\mathbb{A}_0}$ are equivalent. Hence, in view of (2.46), $\mathbb{V}$ is $\mathbb{A}_0$-bounded. Thus, the operator $\mathbb{A}=\mathbb{A}_0+\mathbb{V}$, as a $\mathbb{A}_0$-bounded perturbation of the generator $\mathbb{A}_0$ of a C_0-semigroup on $\mathbb{X}$, is the generator of a C_0-semigroup on $\mathbb{X}$ (see Theorem 1.1.7).

(iii) We will prove that $\rho(\mathbb{A})=s_\beta^{-1}(\rho(A))$, which is equivalent to the assertion. To this end, we will show that, given (u,v), $(g,h)\in\mathbb{X}$ and $\mu\in\mathbb{C}$, the condition

$$(u,v)\in D(\mathbb{A})\quad\text{and}\quad(\mu I-\mathbb{A})(u,v)=(g,h)\tag{2.47}$$

is equivalent to

$$u \in H^2(\mathbb{R}^N), \quad v = \mu u - g \ \text{ and } \ (s_\beta(\mu)I - A)u = -h - (\mu+\beta)g. \tag{2.48}$$

Indeed, suppose that (2.47) holds. Then $u \in H^2(\mathbb{R}^N)$ and

$$\mu u - v = g \quad \text{and} \quad \mu v - (-Au - \beta v) = h.$$

Hence, we have

$$v = \mu u - g, \tag{2.49}$$

i.e., the first equality in (2.48), and

$$(\mu+\beta)v + Au = h. \tag{2.50}$$

Consequently, substituting (2.49) in (2.50), we obtain

$$(\mu+\beta)(\mu u - g) + Au = h,$$

which entails the second equality in (2.48).

Next, let (u, v), $(g, h) \in \mathbb{X}$ and $\mu \in \mathbb{C}$ be such that (2.48) is satisfied. Then $u \in H^2(\mathbb{R}^N)$ and $g \in H^1(\mathbb{R}^N)$, thus, the first equality in (2.48) yields $v \in H^1(\mathbb{R}^N)$ and therefore $(u, v) \in D(\mathbb{A})$. From the second equality in (2.48) we deduce that

$$Au + \mu(\mu+\beta)u = h + (\mu+\beta)g.$$

We apply the first equality in (2.48) to obtain

$$\begin{aligned}(\mu I - \mathbb{A})(u, v) &= (\mu u, \mu v) - (v, -Au - \beta v) = (\mu u - v, \mu v + Au + \beta v) \\ &= \big(\mu u - (\mu u - g), Au + (\mu+\beta)(\mu u - g)\big) \\ &= \big(g, Au + \mu(\mu+\beta)u - (\mu+\beta)g\big) = (g, h),\end{aligned}$$

hence, (2.47) is satisfied.

Let $\mu \in \rho(\mathbb{A})$. Then, in particular, for $g = 0$ and any $h \in L^2(\mathbb{R}^N)$, there is a unique $(u, v) \in D(\mathbb{A})$ such that

$$(\mu I - \mathbb{A})(u, v) = (0, -h). \tag{2.51}$$

Conditions (2.47) and (2.48) are equivalent, thus, we get

$$(s_\beta(\mu)I - A)u = h. \tag{2.52}$$

We claim that u is the unique solution of (2.52). Indeed, suppose that $\widetilde{u} \in H^2(\mathbb{R}^N)$ also satisfies (2.52). We define $\widetilde{v} = \mu\widetilde{u}$ and, since (2.47) and (2.48) are equivalent, we obtain that $(\widetilde{u}, \widetilde{v})$ belongs to $D(\mathbb{A})$ and fulfills (2.51). Hence, $\widetilde{u} = u$, and, since A is closed, $s_\beta(\mu) \in \rho(A)$ ([3]).

Conversely, suppose that $s_\beta(\mu) \in \rho(A)$ and we take any $(g, h) \in \mathbb{X}$. Therefore, there exists a unique $u \in H^2(\mathbb{R}^N)$ such that

$$(s_\beta(\mu)I - A)u = -h - (\mu+\beta)g.$$

We put $v = \mu u - g$ and, because (2.47) and (2.48) are equivalent, we get that

$$(u, v) \in D(\mathbb{A}) \quad \text{and} \quad (\mu I - \mathbb{A})(u, v) = (g, h).$$

([3])If $A : D(A) \subset X \to X$ is a closed operator in a Banach space X, then, by the closed graph theorem, $\lambda \in \rho(A)$ if and only if, for any $f \in X$, there exists a unique $u \in D(A)$ such that $\lambda u - Au = f$.

Using the equivalence of the conditions (2.47) and (2.48), we infer that (u, v) is the only solution of the equation above. By (i), the operator $\mathbb{A}$ is closed, thus $\mu \in \rho(\mathbb{A})$.

Further, observe that $\mu \in s_\beta^{-1}(\{\lambda\})$ if and only if $s_\beta(\mu) = \lambda$, which leads to the equation

$$\mu^2 + \beta\mu + \lambda = 0$$

and its solutions are given by the formulas (2.42).

(iv) Let $\mu \in \mathbb{C}$ and assume that $(u, v) \in \operatorname{Ker}(\mu I - \mathbb{A})$. Because (2.47) and (2.48) are equivalent, this implies that $u \in H^2(\mathbb{R}^N)$, $v = \mu u$ and $(s_\beta(\mu)I - A)u = 0$. On the other hand, let $u \in \operatorname{Ker}(s_\beta(\mu)I - A)$ and put $v = \mu u$. From the equivalence of (2.47) and (2.48), we deduce that $(u, v) \in D(\mathbb{A})$ and $(\mu I - \mathbb{A})(u, v) = 0$. Hence, (2.43) holds.

Let $\mu \in \sigma_p(\mathbb{A})$. Then $\operatorname{Ker}(\mu I - \mathbb{A}) \neq \{0\}$, thus, (2.43) entails that there exists $u \neq 0$ such that $u \in \operatorname{Ker}(s_\beta(\mu)I - A)$, i.e., $s_\beta(\mu) \in \sigma_p(A)$. On the other hand, if $s_\beta(\mu) \in \sigma_p(A)$, then $\operatorname{Ker}(s_\beta(\mu)I - A) \neq \{0\}$, thus, there exists a nonzero $u \in H^2(\mathbb{R}^N)$ satisfying $(s_\beta(\mu)I - A)u = 0$. Consequently, from (2.43) we deduce that $(u, \mu u) \in \operatorname{Ker}(\mu I - \mathbb{A})$, hence $\mu \in \sigma_p(\mathbb{A})$. □

2.3 k-set contractivity of the C_0-semigroup

Recall that $A : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the linear operator defined by $A = -\Delta + \boldsymbol{V}$ and $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is the linear operator defined by $\mathbb{A}(u, v) = (v, -Au - \beta v)$ for $(u, v) \in D(\mathbb{A}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In this section we assume that $\varrho(V_\infty) > 0$, i.e., the asymptotic bottom of the $L^\infty(\mathbb{R}^N)$-part of the potential V, denoted by V_∞, is positive (see (7)). Let us define

$$\sigma_P = \sigma(A) \cap (-\infty, 0), \quad \sigma_0 = \sigma(A) \cap \{0\}, \quad \sigma_Q = \sigma(A) \cap (0, +\infty) \tag{2.53}$$

and

$$d = \operatorname{dist}(0, \sigma(A) \cap (0, +\infty)). \tag{2.54}$$

We begin by the spectral decomposition of the space $L^2(\mathbb{R}^N)$.

Proposition 2.3.1. *Let*

$$X_P = \bigoplus_{\lambda \in \sigma_P} \operatorname{Ker}(\lambda I - A), \quad X_0 = \operatorname{Ker} A, \quad X_Q = (X_P \oplus X_0)^\perp$$

where the orthogonal complement is taken in $(L^2(\mathbb{R}^N), \langle \cdot, \cdot \rangle_{L^2})$.

(i) X_P, X_0 *are finite-dimensional subspaces and* $L^2(\mathbb{R}^N)$ *admits the orthogonal decomposition*

$$L^2(\mathbb{R}^N) = X_P \oplus X_0 \oplus X_Q. \tag{2.55}$$

with the corresponding projections P, P_0, Q. *Moreover,* X_j, $j \in \{P, Q, 0\}$, *are invariant with respect to* A, *the restrictions (see p. 3)* $A_j = A|_{X_j}$, $j \in \{P, 0\}$, *are bounded and self-adjoint, the restriction* $A_Q = A|_{X_Q}$ *is self-adjoint, and* $\sigma\left(A|_{X_j}\right) = \sigma_j$, $j \in \{P, Q, 0\}$.

(ii) $H^1(\mathbb{R}^N)$ *is a topological direct sum of closed subspaces:*

$$H^1(\mathbb{R}^N) = X_P \oplus X_0 \oplus X'_Q \tag{2.56}$$

where $X'_Q = H^1(\mathbb{R}^N) \cap X_Q$. *Additionally,* $H^2(\mathbb{R}^N) \cap X_Q$ *is dense in* X'_Q.

(iii) $d > 0$, $\sigma_Q = \sigma(A) \cap [d, +\infty)$, *and, for* $u \in H^2(\mathbb{R}^N) \cap X_Q$, *there holds*

$$\langle Au, u \rangle_{L^2} \geq d\|u\|_{L^2}^2. \tag{2.57}$$

Proof. (i) By Proposition 2.1.10 (ii), A is self-adjoint. From (7) and Proposition 2.1.12 (iii), it follows that $\sigma_P \cup \sigma_0$ consists of a finite number of isolated eigenvalues with finite multiplicities. Hence, X_P and X_0 are finite-dimensional. Further, we infer that σ_P, σ_0 are compact and σ_Q is closed. Then, it suffices to apply the Spectral Theorem for Self-Adjoint Operators (Theorem A.1.6) to obtain the remaining part of the assertion.

(ii) Since the subspaces X_P, X_0, and X_Q are closed in $L^2(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$ is continuously embedded into $L^2(\mathbb{R}^N)$, and $X_P \oplus X_0$ is a subspace of $H^1(\mathbb{R}^N)$, it follows from Proposition A.1.2 (ii) that (2.56) holds and that X_P, X_0, and X'_Q are closed subspaces of $H^1(\mathbb{R}^N)$. Moreover, since $H^2(\mathbb{R}^N)$ is a dense subspace of $H^1(\mathbb{R}^N)$ by Friedrich's Theorem (cf. Theorem A.2.3), it follows that $H^2(\mathbb{R}^N) \cap X_Q$ is dense in X'_Q (see Proposition A.1.2 (iii)).

(iii) If $0 \in \rho(A)$, then $d > 0$, because the resolvent set is open. On the other hand, in view of (7) and Proposition 2.1.12 (i), if $0 \in \sigma(A)$, then it is an isolated point of the spectrum, thus again $d > 0$. Hence, $\sigma(A) \cap [d, +\infty) \subset \sigma_Q$. Conversely, if $\lambda \in \sigma_Q$, then $\lambda > 0$, therefore, in view of (2.54), $\lambda \geq d$ and $\sigma_Q \subset \sigma(A) \cap [d, +\infty)$. This yields $\sigma_Q = \sigma(A) \cap [d, +\infty)$. Next, by (i), $A|_{X_Q}$ is self-adjoint and $\sigma\left(A|_{X_Q}\right) = \sigma_Q$, thus (2.57) is a consequence of the spectral bottom formula for self-adjoint operators (see Theorem A.1.5 (i)). □

Remark 2.3.2. Because $H^1(\mathbb{R}^N)$ splits into a topological direct sum of closed subspaces X_P, X_0, and X'_Q, the restrictions of P, P_0, and Q to $H^1(\mathbb{R}^N)$ are well-defined and continuous projections onto X_P, X_0, and X'_Q, respectively (see Proposition A.1.2 (i)). □

In the following theorem, we define a new scalar product on the space $\mathbb{X}$ that depends on a parameter $s > 0$. In Proposition 2.3.7, we show that there exists an $s \in (0, \beta)$, where $\beta > 0$ is a damping coefficient, such that the operator $\mathbb{A}$ is strongly dissipative on $\mathbb{X}_Q$ with respect to this scalar product. This result enables us to prove that the C_0-semigroup generated by $\mathbb{A}$ is a k-set contraction with respect to the Hausdorff measure of non-compactness; see Theorem 2.3.9.

Theorem 2.3.3. *Let V be a Kato-Rellich type potential with a positive asymptotic bottom, i.e., V satisfies* (5a), (5b), *and* (7). *Then, for any $s > 0$, the map*

$$\langle \cdot, \cdot \rangle_{s,V} : \mathbb{X} \times \mathbb{X} \to \mathbb{R},$$

defined for all (u_1, v_1), $(u_2, v_2) \in \mathbb{X}$ by

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{s,V} = \langle (P+P_0)u_1, (P+P_0)u_2 \rangle_{L^2} + \langle Qu_1, Qu_2 \rangle_Q + \langle v_1 + su_1, v_2 + su_2 \rangle_{L^2}, \quad (2.58)$$

is a scalar product on $\mathbb{X}$, where the mapping $\langle \cdot, \cdot \rangle_Q : X'_Q \times X'_Q \to \mathbb{R}$ is given by

$$\langle u, v \rangle_Q = \int_{\mathbb{R}^N} \nabla u(x) \nabla v(x) + V(x) u(x) v(x)\, dx \quad \textit{for} \;\; u, v \in X'_Q. \quad (2.59)$$

The associated function

$$\| \cdot \|_{s,V} : \mathbb{X} \to [0, +\infty),$$

defined for any $(u, v) \in \mathbb{X}$ by

$$\|(u, v)\|^2_{s,V} = \langle (u, v), (u, v) \rangle_{s,V} = \|(P + P_0)u\|^2_{L^2} + \|Qu\|^2_Q + \|v + su\|^2_{L^2}, \quad (2.60)$$

is a norm on $\mathbb{X}$ that is equivalent to the original norm $\| \cdot \|_{\mathbb{X}}$ (see p. 29).

In particular, $\mathbb{X}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{s,V}$ is a Hilbert space.

To prove this theorem we need the following two lemmata.

Lemma 2.3.4. *Assume that V is a Kato-Rellich type potential with a positive asymptotic bottom, i.e., V satisfies* (5a) *and* (5b), *and additionally* (7). *Then*

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2\, dx \geq d\|u\|^2_{L^2} \quad \textit{for} \;\; u \in X'_Q \quad (2.61)$$

where $d > 0$ is given by (2.54).

Proof. Let $u \in X'_Q$. By Proposition 2.3.1 (ii), there exists a sequence $(u_n)_{n\geq 1}$ in $H^2(\mathbb{R}^N) \cap X_Q$ such that $u_n \xrightarrow{n\to\infty} u$ in $H^1(\mathbb{R}^N)$. Hence, based on Proposition 2.3.1 (iii), we have

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^2 + V(x)|u_n(x)|^2\, dx = \langle A u_n, u_n\rangle_{L^2} \geq d\|u_n\|_{L^2}^2 \quad \text{for } n \geq 1. \tag{2.62}$$

Because $u_n \xrightarrow{n\to\infty} u$ in $H^1(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^2\, dx \xrightarrow{n\to\infty} \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx \quad \text{and} \quad \int_{\mathbb{R}^N} V_\infty(x)|u_n(x)|^2\, dx \xrightarrow{n\to\infty} \int_{\mathbb{R}^N} V_\infty(x)|u(x)|^2\, dx.$$

Using Lemma 2.1.13 (i), we obtain

$$u_n \xrightarrow{n\to\infty} u_0 \quad \text{in } L^{2p/(p-1)}(\mathbb{R}^N).$$

Thus, by Proposition A.2.2 (ii), there exist a subsequence $(u_{n_k})_{k\geq 1}$ and a function $f \in L^{2p/(p-1)}(\mathbb{R}^N)$ such that $u_{n_k}(x) \xrightarrow{k\to\infty} u(x)$, for almost every $x \in \mathbb{R}^N$, and

$$|u_{n_k}(x)| \leq f(x) \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \text{almost every } x \in \mathbb{R}^N.$$

We have $V_0\,|f|^2 \in L^1(\mathbb{R}^N)$ by the Hölder inequality, hence the Lebesgue Dominated Convergence Theorem yields

$$\int_{\mathbb{R}^N} V_0(x)|u_{n_k}(x)|^2\, dx \xrightarrow{k\to\infty} \int_{\mathbb{R}^N} V_0(x)|u(x)|^2\, dx.$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla u_{n_k}(x)|^2 + V(x)|u_{n_k}(x)|^2\, dx \xrightarrow{k\to\infty} \int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)|u(x)|^2\, dx,$$

which together with (2.62) completes the proof. □

Lemma 2.3.5. *Let V be a Kato-Rellich type potential with a positive asymptotic bottom, i.e., V satisfies* (5a) *and* (5b)*, and additionally* (7)*. The mapping $\langle\cdot,\cdot\rangle_Q$ given by* (2.59) *is a scalar product on X'_Q and the associated norm $\|u\|_Q = \sqrt{\langle u, u\rangle_Q}$ is equivalent to $\|\cdot\|_{H^1}$ on X'_Q.*

Proof. Clearly, $\langle\cdot,\cdot\rangle_Q$ is bilinear and symmetric. Furthermore, due to Lemma 2.3.4, $\langle u, u\rangle_Q \geq 0$ for $u \in X'_Q$ and $\langle u, u\rangle_Q = 0$ if and only if $u = 0$. Hence, $\langle\cdot,\cdot\rangle_Q$ is a scalar product and consequently $\|\cdot\|_Q$ is a norm.

In view of Lemma 2.1.13 (i), there exists a constant $C_1 > 0$ such that

$$\begin{aligned}\|u\|_Q^2 &\leq \|\nabla u\|_{L^2}^2 + C_1^2\,\|V_0\|_{L^p}\,\|u\|_{H^1}^2 + \|V_\infty\|_{L^\infty}\,\|u\|_{L^2}^2 \\ &\leq (1 + C_1^2\,\|V_0\|_{L^p} + \|V_\infty\|_{L^\infty})\|u\|_{H^1}^2 \quad \text{for} \quad u \in X'_Q.\end{aligned}$$

On the other hand, by Lemma 2.1.14, there exists a constant $C_2 > 0$ such that

$$\|u\|_Q^2 \geq (1/2)\|\nabla u\|_{L^2}^2 - C_2\|u\|_{L^2}^2 \quad \text{for } u \in X'_Q,$$

which, by use of Lemma 2.3.4, gives

$$\begin{aligned}\|u\|_{H^1}^2 &\leq 2\|u\|_Q^2 + (2C_2 + 1)\|u\|_{L^2}^2 \\ &\leq 2\|u\|_Q^2 + \frac{2C_2 + 1}{d}\,\|u\|_Q^2 = \left(2 + \frac{2C_2 + 1}{d}\right)\|u\|_Q^2 \quad \text{for} \quad u \in X'_Q.\end{aligned}$$

□

Proof of Theorem 2.3.3. We readily see that $\langle \cdot, \cdot \rangle_{s,V}$ is bilinear and symmetric. It is also clear that, by Lemma 2.3.5, $\langle (u,v),(u,v)\rangle_{s,V} \geq 0$, for $(u,v) \in \mathbb{X}$, and $\langle (u,v),(u,v)\rangle_{s,V} = 0$ if and only if $(u,v) = (0,0)$. Hence, $\langle \cdot, \cdot \rangle_{s,V}$ is a scalar product and $\| \cdot \|_{s,V}$ is a norm.

Observe that $\|v + su\|_{L^2}^2 \leq 2\|v\|_{L^2}^2 + 2s^2\|u\|_{H^1}^2$ and $\|(P+P_0)u\|_{L^2}^2 \leq \|u\|_{H^1}^2$, for $u \in H^1(\mathbb{R}^N)$ and $v \in L^2(\mathbb{R}^N)$. Next, the norms $\|\cdot\|_Q$ and $\|\cdot\|_{H^1}$ are equivalent, thus, there exist constants C_1, $C_2 > 0$ such that

$$\|Qu\|_Q \leq C_1\|Qu\|_{H^1} \quad \text{and} \quad \|Qu\|_{H^1} \leq C_2\|Qu\|_Q \quad \text{for } u \in H^1(\mathbb{R}^N),$$

see Lemma 2.3.5. Consequently, due to the boundedness of $Q|_{H^1(\mathbb{R}^N)}$, we get

$$\|Qu\|_Q^2 \leq C_1^2 \, \|Qu\|_{H^1}^2 \leq C_1^2 \, \|Q\|_{\mathcal{L}(H^1)}^2 \, \|u\|_{H^1}^2 \quad \text{for } u \in H^1(\mathbb{R}^N).$$

Therefore, there exists $K_1 > 0$ such that, for any $(u,v) \in \mathbb{X}$, $\|(u,v)\|_{s,V} \leq K_1\|(u,v)\|_{\mathbb{X}}$.

The subspace $X_P \oplus X_0$ is finite-dimensional, thus, any two norms are equivalent, in particular, there exists a constant $C_3 > 0$ such that

$$\|(P+P_0)u\|_{H^1} \leq C_3\|(P+P_0)u\|_{L^2} \quad \text{for } u \in H^1(\mathbb{R}^N).$$

This yields

$$\|u\|_{H^1}^2 \leq 2\|Qu\|_{H^1}^2 + 2\|(P+P_0)u\|_{H^1}^2 \leq 2C_2^2\|Qu\|_Q^2 + 2C_3^2\|(P+P_0)u\|_{L^2}^2 \quad \text{for } u \in H^1(\mathbb{R}^N).$$

Further, in the light of Lemma 2.3.4, one has

$$\begin{aligned}\|v\|_{L^2}^2 &\leq 2\|v+su\|_{L^2}^2 + 2s^2\|u\|_{L^2}^2 = 2\|v+su\|_{L^2}^2 + 2s^2\|(P+P_0)u\|_{L^2}^2 + 2s^2\|Qu\|_{L^2}^2 \\ &\leq 2\|v+su\|_{L^2}^2 + 2s^2\|(P+P_0)u\|_{L^2}^2 + \frac{2s^2}{d}\,\|Qu\|_Q^2,\end{aligned}$$

for $u \in H^1(\mathbb{R}^N)$ and $v \in L^2(\mathbb{R}^N)$. Hence, there exists a constant $K_2 > 0$ such that, for all $(u,v) \in \mathbb{X}$, $\|(u,v)\|_{\mathbb{X}} \leq K_2\|(u,v)\|_{s,V}$. □

Now, using the decomposition of the space $L^2(\mathbb{R}^N)$ from Proposition 2.3.1, we provide decomposition of the space $\mathbb{X}$ equipped with the scalar product $\langle \cdot,\cdot\rangle_{\mathbb{X}}$ and the corresponding norm $\|\cdot\|_{\mathbb{X}}$.

Proposition 2.3.6. *Let $\mathbb{X}_P = X_P \times X_P$, $\mathbb{X}_0 = X_0 \times X_0$, and $\mathbb{X}_Q = X_Q' \times X_Q$ be the spaces equipped with the induced scalar product $\langle \cdot,\cdot\rangle_{\mathbb{X}}$ and the norm $\|\cdot\|_{\mathbb{X}}$.*

(i) The space $\mathbb{X}$ decomposes as a topological direct sum

$$\mathbb{X} = \mathbb{X}_P \oplus \mathbb{X}_0 \oplus \mathbb{X}_Q \tag{2.63}$$

with the corresponding projections $\mathbb{P}$, $\mathbb{P}_0$, and $\mathbb{Q}$. These projections are bounded linear operators and, for any $s > 0$ and a Kato-Rellich type potential V with a positive asymptotic bottom, the decomposition (2.63) is orthogonal with respect to the scalar product $\langle \cdot,\cdot\rangle_{s,V}$.

(ii) The spaces $\mathbb{X}_P$, $\mathbb{X}_0$, and $\mathbb{X}_Q$ are invariant under $\mathbb{A}$. Moreover, the restrictions $\mathbb{A}|_{\mathbb{X}_P}$, $\mathbb{A}|_{\mathbb{X}_0}$ are bounded, and the restriction $\mathbb{A}|_{\mathbb{X}_Q}$ is densely defined and closed operator.

(iii) For any $t \geq 0$, the subspaces $\mathbb{X}_P$, $\mathbb{X}_0$, $\mathbb{X}_Q$ are invariant under $e^{t\mathbb{A}}$, and $e^{t\mathbb{A}}$ commutes with the projections $\mathbb{P}$, $\mathbb{P}_0$, $\mathbb{Q}$. Furthermore, the restrictions $\mathbb{A}|_{\mathbb{X}_P}$, $\mathbb{A}|_{\mathbb{X}_0}$, $\mathbb{A}|_{\mathbb{X}_Q}$ are the generators of C_0-semigroups on the subspaces $\mathbb{X}_P$, $\mathbb{X}_0$, $\mathbb{X}_Q$, respectively, and

$$e^{t\mathbb{A}}\big|_{\mathbb{X}_P} = e^{t\mathbb{A}|_{\mathbb{X}_P}}, \quad e^{t\mathbb{A}}\big|_{\mathbb{X}_0} = e^{t\mathbb{A}|_{\mathbb{X}_0}}, \quad e^{t\mathbb{A}}\big|_{\mathbb{X}_Q} = e^{t\mathbb{A}|_{\mathbb{X}_Q}}, \quad \textit{for any } t \geq 0. \tag{2.64}$$

Proof. (i) Let $(u, v) \in \mathbb{X}$. Recall that $L^2(\mathbb{R}^N)$ splits into an orthogonal sum of subspaces X_P, X_0, X_Q, and $H^1(\mathbb{R}^N)$ splits into a topological direct sum of closed subspaces X_P, X_0, X'_Q (Proposition 2.3.1 (i) and (ii)). Thus, there are the unique $u_P \in X_P$, $u_0 \in X_0$, $u_Q \in X'_Q$ such that $u = u_P + u_0 + u_Q$ and the unique $v_P \in X_P$, $v_0 \in X_0$, $v_Q \in X_Q$ such that $v = v_P + v_0 + v_Q$. Therefore, we obtain the unique decomposition $(u, v) = (u_P, v_P) + (u_0, v_0) + (u_Q, v_Q)$. Hence, $\mathbb{X}$ is a direct sum of subspaces $\mathbb{X}_P$, $\mathbb{X}_0$, and $\mathbb{X}_Q$. Since, for any sequence $\big((u_n, v_n)\big)_{n\geq 1}$ in $\mathbb{X}$, $(u_n, v_n) \xrightarrow{n\to\infty} (u_0, v_0)$ in $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ if and only if $u_n \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^N)$ and $v_n \xrightarrow{n\to\infty} v_0$ in $L^2(\mathbb{R}^N)$, it is easily seen that $\mathbb{X}_P$, $\mathbb{X}_0$, $\mathbb{X}_Q$ are closed subspaces. Therefore, Proposition A.1.2 (i) implies that the projections $\mathbb{P}$, $\mathbb{P}_0$, $\mathbb{Q}$ are bounded linear operators.

Further, let $(u, v) \in \mathbb{X}_P \oplus \mathbb{X}_0$ and $(u_Q, v_Q) \in \mathbb{X}_Q$. Then, $(P + P_0)u_Q = 0$, $Qu = 0$ and $v + su \perp v_Q + su_Q$ in $L^2(\mathbb{R}^N)$, thus

$$\langle (u, v), (u_Q, v_Q)\rangle_{s,V} = \langle (P + P_0)u, (P + P_0)u_Q\rangle_{L^2} + \langle Qu, Qu_Q\rangle_Q + \langle v + su, v_Q + su_Q\rangle_{L^2} = 0.$$

Hence, $\mathbb{X}_Q \perp \mathbb{X}_P \oplus \mathbb{X}_0$ in $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$. Now, let $(u_P, v_P) \in \mathbb{X}_P$ and $(u_0, v_0) \in \mathbb{X}_0$. The subspace X_P is perpendicular to the subspace X_0 in $L^2(\mathbb{R}^N)$, therefore $\langle (u_P, v_P), (u_0, v_0)\rangle_{s,V} = 0$ and $\mathbb{X}_P \perp \mathbb{X}_0$ in $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$.

(ii) By Proposition 2.3.1 (i), subspaces X_P, X_0, and X_Q are invariant under A. Hence, taking into account the formula for $\mathbb{A}$, we see that $\mathbb{X}_P$, $\mathbb{X}_0$, and $\mathbb{X}_Q$ are invariant under $\mathbb{A}$. Note that the restrictions $\mathbb{A}|_{\mathbb{X}_P}$ and $\mathbb{A}|_{\mathbb{X}_0}$ are bounded, because they are the linear maps on the finite-dimensional spaces $\mathbb{X}_P$ and $\mathbb{X}_0$, respectively.

Obviously, $\mathbb{X}_P \oplus \mathbb{X}_0$ is a subspace of $D(\mathbb{A})$. Moreover, due to Proposition 2.2.1 (i), $D(\mathbb{A})$ is a dense subspace of $\mathbb{X}$. Then, based on Proposition A.1.2 (iii), we obtain

$$\overline{D(\mathbb{A}) \cap \mathbb{X}_Q}^{\mathbb{X}} = \overline{D(\mathbb{A})}^{\mathbb{X}} \cap \mathbb{X}_Q = \mathbb{X}_Q.$$

Let us remind that the operator $\mathbb{A}$ is closed, see Proposition 2.2.1 (i). On the other hand,

$$\mathrm{Gr}\ \mathbb{A}|_{\mathbb{X}_Q} = \mathrm{Gr}\,\mathbb{A} \cap (\mathbb{X}_Q \times \mathbb{X}_Q).$$

Hence, Gr $\mathbb{A}|_{\mathbb{X}_Q}$ is closed, i.e., the operator $\mathbb{A}|_{\mathbb{X}_Q}$ is closed.

(iii) Proposition 2.2.1 (ii) shows that $\mathbb{A}$ generates a C_0-semigroup. Furthermore, in view of (ii), $\mathbb{A}(\mathbb{X}_P) \subset \mathbb{X}_P$, $\mathbb{A}(\mathbb{X}_0) \subset \mathbb{X}_0$ and $\mathbb{A}(D(\mathbb{A}) \cap \mathbb{X}_Q) \subset \mathbb{X}_Q$. Then from Proposition 1.1.9 we deduce that, for any $t \geq 0$, subspaces $\mathbb{X}_P$, $\mathbb{X}_0$, $\mathbb{X}_Q$ are invariant with respect to $e^{t\mathbb{A}}$, $e^{t\mathbb{A}}$ commutes with projections $\mathbb{P}$, $\mathbb{P}_0$, $\mathbb{Q}$, linear operators $\mathbb{A}|_{\mathbb{X}_P}$, $\mathbb{A}|_{\mathbb{X}_0}$, $\mathbb{A}|_{\mathbb{X}_Q}$ are the generators of C_0-semigroups, and the formulae (2.64) are fulfilled. □

After defining the equivalent norm $\|\cdot\|_{s,V}$ on the space $\mathbb{X}$ and providing the decomposition of $\mathbb{X}$, we are going to prove that the C_0-semigroup generated by the damped wave operator $\mathbb{A}$ is a k-set contraction with respect to the Hausdorff measure of non-compactness on $(\mathbb{X}, \|\cdot\|_{s,V})$ for a properly chosen $s > 0$.

Proposition 2.3.7. *There are $s > 0$ and $\rho > 0$ such that*

$$\langle \mathbb{A}(u, v), (u, v)\rangle_{s,V} \leq -\rho\|(u, v)\|^2_{s,V} \ \text{ for all } (u, v) \in D(\mathbb{A}) \cap \mathbb{X}_Q. \tag{2.65}$$

Proof. We note that, for $(u, v) \in D(\mathbb{A}) \cap \mathbb{X}_Q$, i.e. $u \in H^2(\mathbb{R}^N) \cap X_Q$ and $v \in X'_Q$,

$$\langle Au, v\rangle_{L^2} = \langle -\Delta u + \boldsymbol{V} u, v\rangle_{L^2} = \langle \nabla u, \nabla v\rangle_{L^2} + \langle \boldsymbol{V} u, v\rangle_{L^2} = \langle u, v\rangle_Q, \tag{2.66}$$

thus, in particular, for $u \in H^2(\mathbb{R}^N) \cap X_Q$,

$$\langle Au, u\rangle_{L^2} = \|u\|^2_Q. \tag{2.67}$$

Taking into account (2.66) and (2.67), by a direct computation, for any $s \in (0, \beta)$ and $(u, v) \in D(\mathbb{A}) \cap \mathbb{X}_Q$, one has

$$\begin{aligned}
\langle \mathbb{A}(u,v),(u,v)\rangle_{s,V} &= \langle (v, -Au - \beta v),(u,v)\rangle_{s,V} \\
&= \langle v,u\rangle_Q + \langle -Au - \beta v + sv, v + su\rangle_{L^2} \\
&= \langle v,u\rangle_Q + \langle -Au - (\beta - s)v, v + su\rangle_{L^2} \\
&= \langle v,u\rangle_Q - \langle Au, v\rangle_{L^2} - s\langle Au,u\rangle_{L^2} - (\beta - s)\langle v, v+su\rangle_{L^2} \\
&= -s\|u\|_Q^2 - (\beta - s)\|v + su\|_{L^2}^2 + s(\beta - s)\langle u, v + su\rangle_{L^2}.
\end{aligned}$$

Further, by use of Young's inequality with an arbitrary $\varepsilon > 0$ (see Lemma A.2.1 (i)), and next Proposition 2.3.1 (iii),

$$\begin{aligned}
\langle \mathbb{A}(u,v),(u,v)\rangle_{s,V} &\leq -s\|u\|_Q^2 - (\beta - s)\|v + su\|_{L^2}^2 + s(\beta - s)\left(\frac{\|u\|_{L^2}^2}{4\varepsilon} + \varepsilon\|v + su\|_{L^2}^2\right) \\
&\leq -s\|u\|_Q^2 - (\beta - s)\|v + su\|_{L^2}^2 + s(\beta - s)\left(\frac{\|u\|_Q^2}{4d\varepsilon} + \varepsilon\|v + su\|_{L^2}^2\right) \qquad (2.68) \\
&= -s\left(1 - \frac{\beta - s}{4d\varepsilon}\right)\|u\|_Q^2 - (\beta - s)(1 - s\varepsilon)\|v + su\|_{L^2}^2.
\end{aligned}$$

If we find $s \in (0, \beta)$ and $\varepsilon > 0$ such that

$$1 - \frac{\beta - s}{4d\varepsilon} > 0 \quad \text{and} \quad 1 - s\varepsilon > 0, \tag{2.69}$$

then, for

$$\rho = \min\left\{s\left(1 - \frac{\beta - s}{4d\varepsilon}\right), (\beta - s)(1 - s\varepsilon)\right\}, \tag{2.70}$$

we shall obtain

$$\langle \mathbb{A}(u,v),(u,v)\rangle_{s,V} \leq -\rho(\|u\|_Q^2 + \|v + su\|_{L^2}^2) = -\rho\|(u,v)\|_{s,V}^2. \tag{2.71}$$

Indeed, (2.69) is equivalent to the inequalities

$$\frac{1}{s} > \varepsilon > \frac{\beta - s}{4d} \tag{2.72}$$

and one can easily check that, for sufficiently small $s \in (0, \beta)$, $1/s > (\beta - s)/4d$ and as ε we can choose any number from the interval $((\beta - s)/4d, 1/s)$. This completes the proof. □

Corollary 2.3.8. *We define $s_0 \in (0, \beta/2]$,*

$$s_0 = \begin{cases} \beta/2 & \text{if } d \geq \dfrac{(\beta/2)^2}{4}, \\ \beta/2 - \sqrt{(\beta/2)^2 - 4d} & \text{if } d < \dfrac{(\beta/2)^2}{4}, \end{cases}$$

and function $\eta : (0, \beta/2) \to \mathbb{R}$,

$$\eta(s) = \frac{1 - \dfrac{s(\beta - s)}{4d}}{1 + \dfrac{s^2(\beta - s)}{4d(\beta - 2s)}}.$$

Then, for any number $s \in (0, s_0)$, i.e., s fulfilling $s(\beta - s) < 4d$, the number $\rho > 0$ given by $\rho = s\,\eta(s)$ is such that the inequality (2.65) from Proposition 2.3.7 is satisfied.

Proof. Let $s \in (0, s_0)$ and define number $c \in (0,1)$ by

$$c = \frac{s(\beta - s)}{4d}.$$

In view of (2.72), one may represent number ε as

$$\varepsilon = \frac{1-\lambda}{s} + \frac{\lambda(\beta - s)}{4d}$$

for some $\lambda \in (0,1)$. We have

$$\varepsilon = \frac{1-\lambda}{s} + \frac{\frac{\lambda s(\beta - s)}{4d}}{s} = \frac{1-\lambda}{s} + \frac{\lambda c}{s} = \frac{1-\lambda+\lambda c}{s}.$$

Then we get

$$\begin{aligned} s\left(1 - \frac{\beta - s}{4d\varepsilon}\right) &= s - \frac{c}{\varepsilon} = s - \frac{cs}{1-\lambda+\lambda c} = s\left(1 - \frac{c}{1-\lambda+\lambda c}\right) \\ &= s \cdot \frac{1-\lambda+\lambda c - c}{1-\lambda+\lambda c} = s \cdot \frac{(1-\lambda)(1-c)}{1-\lambda+\lambda c} = s \cdot \frac{1-c}{1 + \frac{\lambda}{1-\lambda} \cdot c} \end{aligned} \tag{2.73}$$

and

$$(\beta - s)(1 - s\varepsilon) = (\beta - s)(1 - (1 - \lambda + \lambda c)) = (\beta - s)(\lambda - \lambda c) = \lambda(1-c)(\beta - s).$$

This implies

$$\frac{s\left(1 - \frac{\beta - s}{4d\varepsilon}\right)}{(\beta - s)(1 - s\varepsilon)} = \frac{s \cdot \frac{1-c}{1 + \frac{\lambda}{1-\lambda} \cdot c}}{\lambda(1-c)(\beta - s)} = \frac{s \cdot \frac{1}{1 + \frac{\lambda}{1-\lambda} \cdot c}}{\lambda(\beta - s)} < \frac{s}{\lambda(\beta - s)}$$

Putting $\lambda = s/(\beta - s)$ we obtain that

$$s\left(1 - \frac{\beta - s}{4d\varepsilon}\right) < (\beta - s)(1 - s\varepsilon).$$

Let us note that, since $0 < s < s_0 \leq \beta/2$, one has indeed $0 < \lambda < 1$, where $\lambda = s/(\beta - s)$. Further, observe that

$$\frac{\lambda}{1-\lambda} \cdot c = \frac{\frac{s}{\beta - s}}{1 - \frac{s}{\beta - s}} \cdot \frac{s(\beta - s)}{4d} = \frac{s^2}{\left(1 - \frac{s}{\beta - s}\right) 4d} = \frac{s^2(\beta - s)}{4d(\beta - 2s)}.$$

Thus, by formula (2.70) for number ρ and (2.73), we have

$$\rho = s\left(1 - \frac{\beta - s}{4d\varepsilon}\right) = s \cdot \frac{1-c}{1 + \frac{\lambda}{1-\lambda} \cdot c} = s \cdot \frac{1 - \frac{s(\beta - s)}{4d}}{1 + \frac{s^2(\beta - s)}{4d(\beta - 2s)}} = s\,\eta(s). \tag{2.74}$$

The proof is completed. □

We state now the main result of this section.

Theorem 2.3.9. *If $s > 0$ and $\rho > 0$ are as in Proposition 2.3.7, then, for all $(u, v) \in \mathbb{X}_Q$ and all $t > 0$, the following inequality holds*

$$\|e^{t\mathbb{A}}(u, v)\|_{s,V} \leq e^{-\rho t}\|(u, v)\|_{s,V} \tag{2.75}$$

and, for any bounded sets $Z \subset \mathbb{X}$ and $D \subset [0, +\infty)$,

$$\chi_{s,V}\big(\{e^{t\mathbb{A}}(u, v) : (u, v) \in Z, t \in D\}\big) \leq e^{-\rho \inf D}\chi_{s,V}(Z). \tag{2.76}$$

In particular, for any bounded $Z \subset \mathbb{X}$ and $t > 0$,

$$\chi_{s,V}(e^{t\mathbb{A}}(Z)) \leq e^{-\rho t}\chi_{s,V}(Z).$$

Proof. Taking $(u, v) \in D(\mathbb{A}) \cap \mathbb{X}_Q$, Proposition 2.3.7 entails, for all $t > 0$,

$$\begin{aligned}\frac{d}{dt}\|e^{t\mathbb{A}}(u, v)\|^2_{s,V} &= \left\langle \frac{d}{dt}e^{t\mathbb{A}}(u, v), e^{t\mathbb{A}}(u, v)\right\rangle_{s,V} + \left\langle e^{t\mathbb{A}}(u, v), \frac{d}{dt}e^{t\mathbb{A}}(u, v)\right\rangle_{s,V} \\ &= 2\left\langle \mathbb{A}e^{t\mathbb{A}}(u, v), e^{t\mathbb{A}}(u, v)\right\rangle_{s,V} \leq -2\rho\|e^{t\mathbb{A}}(u, v)\|^2_{s,V}.\end{aligned}$$

Hence, we obtain

$$e^{2\rho t}\,\frac{d}{dt}\|e^{t\mathbb{A}}(u, v)\|^2_{s,V} \leq -2\,\rho\, e^{2\rho t}\,\|e^{t\mathbb{A}}(u, v)\|^2_{s,V}$$

and, as a consequence,

$$e^{2\rho t}\,\frac{d}{dt}\|e^{t\mathbb{A}}(u, v)\|^2_{s,V} + \frac{d}{dt}(e^{2\rho t})\,\|e^{t\mathbb{A}}(u, v)\|^2_{s,V} \leq 0.$$

Therefore, we arrive at

$$\frac{d}{dt}\big(e^{2\rho t}\|e^{t\mathbb{A}}(u, v)\|^2_{s,V}\big) \leq 0.$$

We integrate the inequality above from 0 to t and get

$$e^{2\rho t}\|e^{t\mathbb{A}}(u, v)\|^2_{s,V} - \|(u, v)\|^2_{s,V} \leq 0,$$

which gives (2.75) for $(u, v) \in D(\mathbb{A}) \cap \mathbb{X}_Q$.

In view of Proposition 2.3.6 (ii), $D(\mathbb{A}) \cap \mathbb{X}_Q$ is dense in $\mathbb{X}_Q$ and, since, for fixed $t \geq 0$, $e^{t\mathbb{A}}$ is a bounded linear operator, (2.75) holds for all $(u, v) \in \mathbb{X}_Q$.

Put

$$W = \{e^{t\mathbb{A}}(u, v) : t \in D, (u, v) \in Z\}.$$

Since $\mathbb{R}$ and $\mathbb{X}$ are separable metric spaces, there exist countable and dense subsets $D_0 \subset D$ and $Z_0 \subset Z$. Let us denote

$$W_0 = \{e^{t\mathbb{A}}(u, v) : t \in D_0, (u, v) \in Z_0\}.$$

We claim that $W \subset \overline{W}_0$. Indeed, let $t \in D$ and $(u, v) \in Z$. There are sequences $(t_n)_{n\geq 1}$ in D_0 and $\big((u_n, v_n)\big)_{n\geq 1}$ in Z_0 such that $t_n \xrightarrow{n\to\infty} t$ and $(u_n, v_n) \xrightarrow{n\to\infty} (u, v)$. This yields $e^{t\mathbb{A}}(u, v) = \lim_{n\to\infty} e^{t_n\mathbb{A}}(u_n, v_n)$, i.e. $e^{t\mathbb{A}}(u, v) \in \overline{W}_0$. Hence, $W_0 \subset W \subset \overline{W}_0$, and, applying the monotonicity of the Hausdorff measure of non-compactness (Proposition A.3.1 (vi)) and the invariance under passage to the closure (Proposition A.3.1 (ii)), we get that

$$\chi_{s,V}(W) = \chi_{s,V}(W_0).$$

Let sequences $(t_n)_{n\geq 1}$ and $\big((u_n, v_n)\big)_{n\geq 1}$ be such that

$$W_0 = \{e^{t_n\mathbb{A}}(u_n, v_n)\}_{n\geq 1}.$$

From the algebraic semi-additivity of the measure of non-compactness (Proposition A.3.1 (iii)) it follows that

$$\chi_{s,V}\big(\{e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big) \leq \chi_{s,V}\big(\{(\mathbb{P}+\mathbb{P}_0)e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big) + \chi_{s,V}\big(\{\mathbb{Q}e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big).$$

We see that $\{(\mathbb{P}+\mathbb{P}_0)e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}$ is a bounded subset of the finite-dimensional subspace $\mathbb{X}_P \oplus \mathbb{X}_0$, thus it is relatively compact in $\mathbb{X}$. Therefore,

$$\chi_{s,V}\big(\{(\mathbb{P}+\mathbb{P}_0)e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big) = 0$$

and

$$\chi_{s,V}\big(\{e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big) \leq \chi_{s,V}\big(\{\mathbb{Q}e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big).$$

By Proposition 2.3.6 (i), $\mathbb{Q}$ is an orthogonal projection in $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$, thus, $\|\mathbb{Q}\|_{\mathcal{L}(\mathbb{X},\|\cdot\|_{s,V})} = 1$, and, by Proposition 2.3.6 (iii), $e^{t\mathbb{A}}$ commutes with $\mathbb{Q}$. Hence, due to (2.75), for $t > 0$ and $(u,v) \in \mathbb{X}$, we get

$$\|\mathbb{Q}e^{t\mathbb{A}}(u,v)\|_{s,V} = \|e^{t\mathbb{A}}\mathbb{Q}(u,v)\|_{s,V} \leq e^{-\rho t}\|\mathbb{Q}(u,v)\|_{s,V} \leq e^{-\rho t}\|(u,v)\|_{s,V}.$$

Thus, $\|\mathbb{Q}e^{t\mathbb{A}}\|_{\mathcal{L}(\mathbb{X},\|\cdot\|_{s,V})} \leq e^{-\rho t}$. Moreover, note that, because D is bounded and $\{e^{t\mathbb{A}}\}_{t\geq 0}$ is a C_0-semigroup, the set $\{e^{t_n\mathbb{A}}(u,v)\}_{n\geq 1}$ is relatively compact for any fixed $(u,v) \in \mathbb{X}$. Then, based on Lemma A.3.2,

$$\chi_{s,V}\big(\{\mathbb{Q}e^{t_n\mathbb{A}}(u_n,v_n)\}_{n\geq 1}\big) \leq \Big(\limsup_{n\to\infty} \|\mathbb{Q}e^{t_n\mathbb{A}}\|_{\mathcal{L}(\mathbb{X},\|\cdot\|_{s,V})}\Big)\big(\chi_{s,V}(\{(u_n,v_n)\}_{n\geq 1})\big) \leq e^{-\rho \inf D}\chi_{s,V}(Z).$$

The proof is completed. □

2.4 Linear topological index formula

To compute the topological index of the operator $e^{t\mathbb{A}}$, in the following lemma we investigate the spectrum of $e^{t\mathbb{A}}$ restricted to the subspaces $\mathbb{X}_P$ and $\mathbb{X}_Q$.

Lemma 2.4.1.

(i) The space $\mathbb{X}_P$ decomposes as a direct sum $\mathbb{X}_P = \mathbb{X}_P^- \oplus \mathbb{X}_P^+$, where

$$\mathbb{X}_P^\pm = \bigoplus_{\lambda\in\sigma_P} \operatorname{Ker}(\mu_\pm(\lambda)I - \mathbb{A})$$

where $\sigma_P = \sigma(A) \cap (-\infty, 0)$ (cf. (2.53)) and numbers $\mu_\pm(\lambda)$ are given by (2.42).

(ii) For any $t > 0$ we have inclusions $e^{t\mathbb{A}}(\mathbb{X}_P^\pm) \subset \mathbb{X}_P^\pm$. Moreover, we have

$$\sigma\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^\pm}\right) = \sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^\pm}\right) = \{e^{\mu_\pm(\lambda)t} : \lambda \in \sigma_P\} \quad \text{for } t > 0, \tag{2.77}$$

and

$$m_{\mathrm{alg}}(e^{\mu_\pm(\lambda)t}) = \dim \operatorname{Ker}(\lambda I - A)$$

for all $t > 0$ and $\lambda \in \sigma_P$, where $m_{\mathrm{alg}}(e^{\mu_\pm(\lambda)t})$ denotes the algebraic multiplicity of the eigenvalue of the operator $e^{t\mathbb{A}}|_{\mathbb{X}_P^\pm}$. In particular, $1 \notin \sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^\pm}\right)$.

(iii) For $t > 0$, $\sigma\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right) \cap \mathbb{R} \subset (-1,1)$, in particular $\sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right) \cap \mathbb{R} \subset (-1,1)$.

(iv) For the subspace $\widetilde{\mathbb{X}}$ *defined by*

$$\widetilde{\mathbb{X}} = \mathbb{X}_P \oplus \mathbb{X}_Q \tag{2.78}$$

we have $1 \in \rho\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right)$ *for* $t > 0$. *Moreover, for* $t > 0$,

$$\operatorname{Ker}(e^{t\mathbb{A}} - I) = \operatorname{Ker}\mathbb{A} = X_0 \times \{0\}.$$

Remark 2.4.2. It follows from Proposition 2.3.6 (i) that the space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ decomposes as a topological direct sum:

$$\mathbb{X} = \mathbb{X}_0 \oplus \widetilde{\mathbb{X}}. \tag{2.79}$$

Thus, by Proposition A.1.2 (i), the operator $\widetilde{\mathbb{P}}$, which is the projection onto $\widetilde{\mathbb{X}}$, is a bounded linear operator. Moreover, by (2.78), we have $\widetilde{\mathbb{P}} = \mathbb{P} + \mathbb{Q}$. Proposition 2.3.6 (i) shows that the decomposition (2.79) is orthogonal in the space $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$. In view of Proposition 2.3.6 (ii), $\widetilde{\mathbb{X}}$ is invariant under $\mathbb{A}$ and, by Proposition 2.3.6 (iii), $\widetilde{\mathbb{X}}$ is invariant under $e^{t\mathbb{A}}$, for any $t \geq 0$.

By $\mathbb{P}_\pm$ we will denote the projections onto $\mathbb{X}_P^\pm$. Observe that, since $\mathbb{X}_P^\pm$ are finite-dimensional, $\mathbb{P}_\pm$ are bounded linear operators. Moreover, by $\chi_{s,V,\widetilde{\mathbb{X}}}$ we will denote the Hausdorff measure of non-compactness in the space $(\widetilde{\mathbb{X}}, \|\cdot\|_{s,V})$. □

Proof of Lemma 2.4.1. (i) Since $\mathbb{X}_P = X_P \times X_P$ and $X_P = \bigoplus_{\lambda\in\sigma_P} \operatorname{Ker}(\lambda I - A)$, we obtain

$$\mathbb{X}_P = \bigoplus_{\lambda\in\sigma_P} \operatorname{Ker}(\lambda I - A) \times \operatorname{Ker}(\lambda I - A).$$

Recall that $\mu_+(\lambda) \neq \mu_-(\lambda)$ for $\lambda \in \sigma_P$, because $\lambda \neq (\beta/2)^2$ (see Proposition 2.2.1 (iii)), and $\operatorname{Ker}(\mu_\pm(\lambda)I - \mathbb{A}) = \{(u, \mu_\pm(\lambda)u) : u \in \operatorname{Ker}(\lambda I - A)\}$ (see Proposition 2.2.1 (iv)). Hence, we have, for all $\lambda \in \sigma_P$,

$$\operatorname{Ker}(\lambda I - A) \times \operatorname{Ker}(\lambda I - A) = \operatorname{Ker}(\mu_-(\lambda)I - \mathbb{A}) \oplus \operatorname{Ker}(\mu_+(\lambda)I - \mathbb{A})$$

and consequently

$$\mathbb{X}_P = \mathbb{X}_P^- \oplus \mathbb{X}_P^+.$$

(ii) Since the finite-dimensional subspaces $\mathbb{X}_P^\pm$ consist of eigenspaces corresponding to the eigenvalues $\mu_\pm(\lambda)$, $\lambda \in \sigma_P$, of $\mathbb{A}$, the operator $\mathbb{A}$ is diagonal on $\mathbb{X}_P^\pm$. Therefore, $\mathbb{X}_P^\pm$ are invariant under $e^{t\mathbb{A}}$, the equalities (2.77) are satisfied and, for any $t > 0$ and $\lambda \in \sigma_P$,

$$m_{\mathrm{alg}}(e^{\mu_\pm(\lambda)t}) = m_{\mathrm{geo}}(e^{\mu_\pm(\lambda)t}) = \dim\operatorname{Ker}(\mu_\pm(\lambda)I - \mathbb{A}).$$

Furthermore, for any $\lambda \in \sigma_P$,

$$\dim\operatorname{Ker}(\mu_\pm(\lambda)I - \mathbb{A}) = \dim\operatorname{Ker}(\lambda I - A).$$

Hence, $m_{\mathrm{alg}}(e^{\mu_\pm(\lambda)t}) = \dim\operatorname{Ker}(\lambda I - A)$ for any $t > 0$ and $\lambda \in \sigma_P$. Since $\sigma_P \subset (-\infty, 0)$, $\mu_-(\lambda) < 0$ and $\mu_+(\lambda) > 0$ (see Proposition 2.2.1 (iii)), therefore $1 \notin \sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^\pm}\right)$.

(iii) By virtue of Theorem 2.3.9,

$$\left\| e^{t\mathbb{A}}|_{\mathbb{X}_Q} \right\|_{\mathcal{L}(\mathbb{X},\|\cdot\|_{s,V})} \leq e^{-\rho t},$$

hence, due to the spectral radius formula (see [52, Thm. VI.6]),

$$\sigma\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right) \subset \overline{B_{\mathbb{C}}(0, e^{-\rho t})}$$

where $\overline{B_{\mathbb{C}}(0, e^{-\rho t})}$ is a closed ball in the complex plane. In particular, $\sigma\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right) \cap \mathbb{R} \subset (-1, 1)$.

(iv) Recall that $\mathbb{X}_P^{\pm}$, $\mathbb{X}_Q$ are invariant under $e^{t\mathbb{A}}$ (see the assertion (ii) and Proposition 2.3.6 (iii)). Hence, we obtain that([4])

$$\rho\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right) = \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^-}\right) \cap \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}\right) \cap \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right).$$

Then, from the point (ii) we deduce that $1 \in \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^-}\right) \cap \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}\right)$ and from the point (iii) we deduce that $1 \in \rho\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right)$. Therefore, $1 \in \rho\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right)$.

Since $1 \in \rho\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right)$, one has $1 \notin \sigma_p\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right)$, which implies

$$\operatorname{Ker}(e^{t\mathbb{A}} - I) = \operatorname{Ker}(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}} - I) \oplus \operatorname{Ker}(e^{t\mathbb{A}}|_{\mathbb{X}_0} - I) = \operatorname{Ker}(e^{t\mathbb{A}}|_{\mathbb{X}_0} - I).$$

Let us note that, as for the subspace $\mathbb{X}_P$, the finite-dimensional subspace $\mathbb{X}_0$ admits the decomposition $\mathbb{X}_0 = \mathbb{X}_0^- \oplus \mathbb{X}_0^+$, where $\mathbb{X}_0^- = \operatorname{Ker}(\mathbb{A} + \beta I)$ and $\mathbb{X}_0^+ = \operatorname{Ker}\mathbb{A}$, hence $\operatorname{Ker}(e^{t\mathbb{A}}|_{\mathbb{X}_0} - I) = \operatorname{Ker}\mathbb{A}$. By Proposition 2.2.1 (iv), $\operatorname{Ker}\mathbb{A} = X_0 \times \{0\}$, and the proof is completed. □

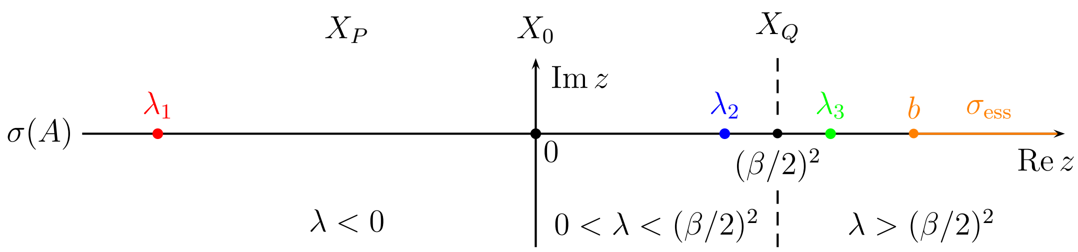
Remark 2.4.3. Consider the Schrödinger operator $A = -\Delta + \boldsymbol{V}$, where $\boldsymbol{V}$ is the operator of multiplication by the Kato-Rellich type potential $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by $V(x) = -a/|x| + b$, with constants a, $b > 0$. By virtue of (2.34), we have

$$\sigma_{\text{disc}}(A) = \left\{\lambda_n = -\frac{a^2}{4n^2} + b : n \geq 1\right\} \quad \text{and} \quad \sigma_{\text{ess}}(A) = [b, +\infty).$$

The pictures below illustrate how the spectrum of the operator $\mathbb{A}$ depends on the spectrum of the operator A. Let us denote $\sigma_{\text{ess}} = \sigma_{\text{ess}}(A)$, and $\mu_{j,\pm} = \mu_{\pm}(\lambda_j)$.

Assume that $a = 2\sqrt{10}$ and $\beta = 8/\sqrt{5}$.

If $b = 5$, then we have $b = 5 > 3.2 = (\beta/2)^2$, and the spectrum of A is as shown below:



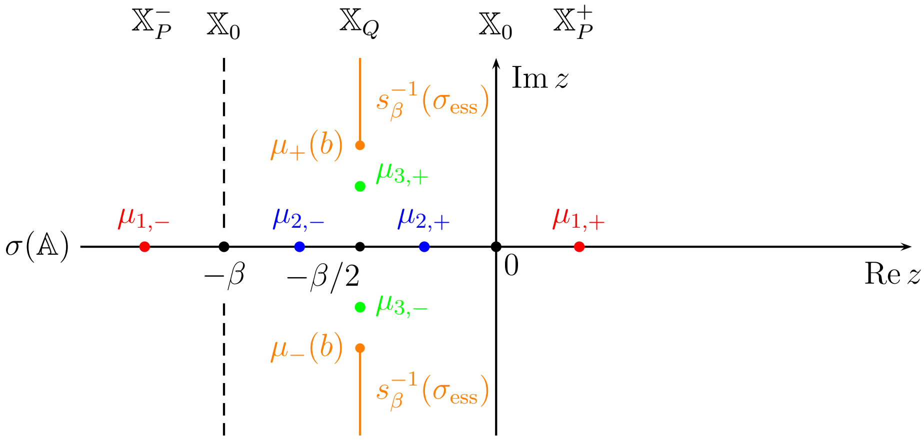
Consequently, the spectrum of the damped wave operator $\mathbb{A}$ is depicted in the following picture:

([4])We apply the following fact. Let X be a Banach space that decomposes as a topological direct sum:

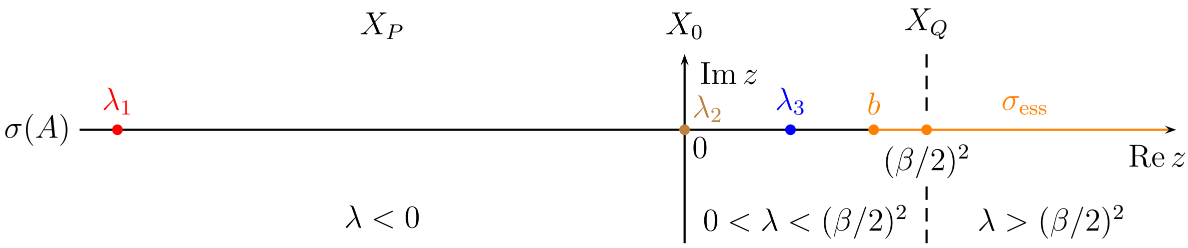
$$X = \bigoplus_{j=1}^{n} X_j.$$

Assume further that $A : D(A) \subset X \to X$ is a linear closed operator such that each subspace X_j is invariant under A for $j = 1, \ldots, n$. Then the following formula holds:

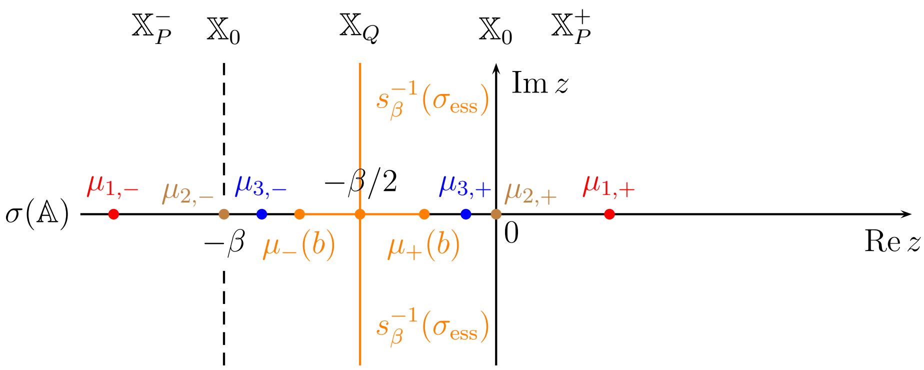
$$\rho(A) = \bigcap_{j=1}^{n} \rho\left(A|_{X_j}\right).$$



If $b = 2.5$, then $b = 2.5 \leq 3.2 = (\beta/2)^2$, which implies that the spectrum of the operator A is as illustrated below:



Accordingly, the spectrum of the operator $\mathbb{A}$ takes the form shown here:



□

Now, we prove the topological index formula for the operator $e^{t\mathbb{A}}$.

Theorem 2.4.4. *Assume that V is a Kato-Rellich type potential such that the asymptotic bottom of its $L^\infty(\mathbb{R}^N)$-part is positive (see conditions* (5a), (5b), *and* (7)*). Moreover, assume that $t > 0$ and $Z \subset \widetilde{\mathbb{X}}$ is an open bounded set such that $(0,0) \in Z$.*

(i) The map $H : \widetilde{\mathbb{X}} \times [0,1] \to \widetilde{\mathbb{X}}$,

$$H(u,v,\mu) = (1-\mu)e^{t\mathbb{A}}(u,v) + \mu e^{t\mathbb{A}}\mathbb{P}_+(u,v)$$

is a well-defined continuous k-set contraction, i.e., for any bounded $W \subset \widetilde{\mathbb{X}}$,

$$\chi_{s,V,\widetilde{\mathbb{X}}}(H(W \times [0,1])) \leq e^{-\rho t}\chi_{s,V,\widetilde{\mathbb{X}}}(W) \tag{2.80}$$

where $s > 0$ and $\rho > 0$ are as in Proposition 2.3.7.

(ii) If $H(u,v,\mu)=(u,v)$ *for some* $(u,v)\in\widetilde{\mathbb{X}}$, $\mu\in[0,1]$, *then* $(u,v)=(0,0)$. *In particular,* $\operatorname{Fix}\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}\right)=\{(0,0)\}$.

(iii) The topological index formula is satisfied:

$$\operatorname{Ind}_C\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}, Z\right)=(-1)^{m_-(A)}$$

where $\operatorname{Ind}_C(\cdot,\cdot)$ *is the topological index for* k*-set contractions in the space* $(\widetilde{\mathbb{X}},\|\cdot\|_{\mathbb{X}})$ *(see Definition A.4.7), and* $m_-(A)$ *is the sum of the multiplicities of the negative eigenvalues of the operator* A.

Proof. (i) By the decompositions $\mathbb{X}=\mathbb{X}_P\oplus\mathbb{X}_0\oplus\mathbb{X}_Q$ and $\mathbb{X}_P=\mathbb{X}_P^-\oplus\mathbb{X}_P^+$ (Proposition 2.3.6 (i) and Lemma 2.4.1 (i)),

$$\widetilde{\mathbb{X}}=\mathbb{X}_P^-\oplus\mathbb{X}_P^+\oplus\mathbb{X}_Q.$$

Hence, we have

$$H(u,v,\mu)=(1-\mu)e^{t\mathbb{A}}\mathbb{Q}(u,v)+(1-\mu)e^{t\mathbb{A}}\mathbb{P}_-(u,v)+e^{t\mathbb{A}}\mathbb{P}_+(u,v). \tag{2.81}$$

In the light of Proposition 2.3.6 (iii), $\mathbb{X}_Q$ is invariant under the operator $e^{t\mathbb{A}}$ and, in the light of Lemma 2.4.1 (ii), $\mathbb{X}_P^\pm$ are invariant under $e^{t\mathbb{A}}$. This shows that the map H is well-defined, i.e. $H(\widetilde{\mathbb{X}}\times[0,1])\subset\widetilde{\mathbb{X}}$. Recall that the projections $\mathbb{Q}$ and $\mathbb{P}_\pm$ are bounded linear operators. Hence, the restrictions of $\mathbb{Q}$, $\mathbb{P}_\pm$ to $\widetilde{\mathbb{X}}$ are also bounded and, therefore, the map H is continuous.

In order to show (2.80) observe that, by the semi-additivity of the Hausdorff measure of non-compactness (Proposition A.3.1 (iii)),

$$\begin{aligned}\chi_{s,V,\widetilde{\mathbb{X}}}(H(W\times[0,1]))\le\chi_{s,V,\widetilde{\mathbb{X}}}&\big(\{(1-\mu)e^{t\mathbb{A}}(u,v)\,:\,(u,v)\in W,\,\mu\in[0,1]\}\big)\\&+\chi_{s,V,\widetilde{\mathbb{X}}}\big(\{\mu e^{t\mathbb{A}}\mathbb{P}_+(u,v)\,:\,(u,v)\in W,\,\mu\in[0,1]\}\big).\end{aligned}\tag{2.82}$$

Using the homogencity of the measure of non-compactness (Proposition A.3.1 (iv)) and its invariance with respect to the subspaces of a Hilbert space (Proposition A.3.1 (viii)), and the k-set contractivity of $\{e^{t\mathbb{A}}\}_{t\ge0}$ (Theorem 2.3.9), we arrive at

$$\begin{aligned}\chi_{s,V,\widetilde{\mathbb{X}}}\big(\{(1-\mu)e^{t\mathbb{A}}(u,v)\,:\,(u,v)\in W,\,\mu\in[0,1]\}\big)&\le\chi_{s,V,\widetilde{\mathbb{X}}}\big(e^{t\mathbb{A}}(W)\big)=\chi_{s,V}\big(e^{t\mathbb{A}}(W)\big)\\&\le e^{-\rho t}\chi_{s,V}(W)=e^{-\rho t}\chi_{s,V,\widetilde{\mathbb{X}}}(W).\end{aligned}\tag{2.83}$$

The set $\{\mu e^{t\mathbb{A}}\mathbb{P}_+(u,v)\,:\,(u,v)\in W,\,\mu\in[0,1]\}$ is bounded and contained in the finite-dimensional space $\mathbb{X}_P^+$, thus

$$\chi_{s,V,\widetilde{\mathbb{X}}}\big(\{\mu e^{t\mathbb{A}}\mathbb{P}_+(u,v)\,:\,(u,v)\in W,\,\mu\in[0,1]\}\big)=0.$$

This together with (2.82) and (2.83) shows (2.80).

(ii) Suppose that $H(u,v,\mu)=(u,v)$. Therefore, using the formula (2.81), one gets

$$\begin{aligned}(1-\mu)e^{t\mathbb{A}}\mathbb{Q}(u,v)&=\mathbb{Q}(u,v),\\(1-\mu)e^{t\mathbb{A}}\mathbb{P}_-(u,v)&=\mathbb{P}_-(u,v),\\e^{t\mathbb{A}}\mathbb{P}_+(u,v)&=\mathbb{P}_+(u,v).\end{aligned}$$

Since $\sigma_P\subset(-\infty,0)$, then, in view of Proposition 2.2.1 (iii), $\mu_-(\lambda)<0$ and $\mu_+(\lambda)>0$, hence, it follows from Lemma 2.4.1 (ii) that

$$\sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^-}\right)\subset(0,1)\quad\text{and}\quad\sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}\right)\subset(1,+\infty). \tag{2.84}$$

In consequence, $\mathbb{P}_+(u,v)=(0,0)$.

For $\mu=1$, we obtain that $\mathbb{Q}(u,v)=(0,0)$ and $\mathbb{P}_-(u,v)=(0,0)$, which implies $(u,v)=(0,0)$.

For $\mu \in [0,1)$, we have

$$e^{t\mathbb{A}}\mathbb{Q}(u,v) = (1-\mu)^{-1}\mathbb{Q}(u,v),$$
$$e^{t\mathbb{A}}\mathbb{P}_-(u,v) = (1-\mu)^{-1}\mathbb{P}_-(u,v).$$

Taking into account (2.84) and the fact that $\sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_Q}\right) \cap \mathbb{R} \subset (-1,1)$ (see Lemma 2.4.1 (iii)), we arrive at $\mathbb{Q}(u,v) = (0,0)$ and $\mathbb{P}_-(u,v) = (0,0)$. Hence, $(u,v) = (0,0)$.

(iii) By virtue of (i), the map H is a k-set contraction in the space $(\widetilde{\mathbb{X}}, \|\cdot\|_{s,V})$, and, by virtue of (ii), $H(u,v,\mu) \neq (u,v)$ for $(u,v) \in \partial Z$ and $\mu \in [0,1]$. Moreover, it follows from Theorem 2.3.3 that $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{s,V}$ are equivalent norms. Therefore, the map H, restricted to $\overline{Z}\times[0,1]$, is an admissible homotopy for the topological index for k-set contractions in $(\widetilde{\mathbb{X}}, \|\cdot\|_{\mathbb{X}})$ (see Definition A.4.7). By the homotopy invariance of the topological index (Theorem A.4.8), we have

$$\mathrm{Ind}_C\left(e^{t\mathbb{A}}|_{\widetilde{\mathbb{X}}}, Z\right) = \mathrm{Ind}_C(H(\cdot,0),Z) = \mathrm{Ind}_C(H(\cdot,1),Z) = \mathrm{Ind}_C\left(\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}}, Z\right).$$

The linear operator $\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ is bounded, thus it transforms bounded sets into bounded sets again. Moreover, $\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}}$ takes values in the finite-dimensional space $\mathbb{X}_P^+$, thus it is compact. This implies that its topological index for k-set contractions coincides with the Leray-Schauder topological index (see Theorem A.4.10):

$$\mathrm{Ind}_C\left(\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}}, Z\right) = \mathrm{Ind}_{LS}\left(\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}}, Z\right).$$

The contraction property of the Leray-Schauder index (Theorem A.4.3) yields

$$\mathrm{Ind}_{LS}\left(\left(e^{t\mathbb{A}}\mathbb{P}_+\right)|_{\widetilde{\mathbb{X}}}, Z\right) = \mathrm{Ind}_{LS}\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}, Z\cap\mathbb{X}_P^+\right)$$

and, since $(0,0) \in Z$, we may use the linear operator index formula (Theorem A.4.4):

$$\mathrm{Ind}_{LS}\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}, Z\cap\mathbb{X}_P^+\right) = (-1)^m \tag{2.85}$$

where m is the sum of algebraic multiplicities of these eigenvalues of $e^{t\mathbb{A}}|_{\mathbb{X}_P^+}$ which are greater than 1. On the other hand, by Lemma 2.4.1 (ii),

$$\sigma_p\left(e^{t\mathbb{A}}|_{\mathbb{X}_P^+}\right) = \{e^{\mu_+(\lambda)t} : \lambda \in \sigma_P\} \subset (1,+\infty)$$

and, for all $\lambda \in \sigma_P$, $m_{\mathrm{alg}}(e^{\mu_+(\lambda)t}) = \dim\mathrm{Ker}(\lambda I - A)$. This implies

$$m = \sum_{\lambda\in\sigma_P} m_{\mathrm{alg}}(e^{\mu_+(\lambda)t}) = \sum_{\lambda\in\sigma(A)\cap(-\infty,0)} \dim\mathrm{Ker}(\lambda I - A) = m_-(A).$$

Combining this with (2.85), we arrive at the desired index formula. □

If we assume that $\mathrm{Ker}\,A = \{0\}$, then $\mathbb{X}_0 = \{(0,0)\}$, so $\mathbb{X} = \widetilde{\mathbb{X}}$. In this case, the Theorem 2.4.4 takes the following form.

Corollary 2.4.5. *Suppose that V is a Kato-Rellich type potential such that the asymptotic bottom of its $L^\infty(\mathbb{R}^N)$-part is positive (see conditions* (5a), (5b), *and* (7)*), and additionally* $\mathrm{Ker}\,A = \{0\}$. *Moreover, suppose that $t > 0$ and $Z \subset \mathbb{X}$ is an open bounded set such that* $(0,0) \in Z$.

(i) The map $H : \mathbb{X}\times[0,1] \to \mathbb{X}$, given as in Theorem 2.4.4 (i), is a well-defined continuous k-set contraction, i.e., for any bounded $W \subset \mathbb{X}$,

$$\chi_{s,V}(H(W\times[0,1])) \leq e^{-\rho t}\chi_{s,V}(W)$$

where $s > 0$ and $\rho > 0$ are as in Proposition 2.3.7.

(ii) *If* $H(u, v, \mu) = (u, v)$ *for some* $(u, v) \in \mathbb{X}$, $\mu \in [0, 1]$, *then* $(u, v) = (0, 0)$. *In particular,* $\mathrm{Fix}(e^{t\mathbb{A}}) = \{(0, 0)\}$.

(iii) *The topological index formula holds:*

$$\mathrm{Ind}_C(e^{t\mathbb{A}}, Z) = (-1)^{m_-(A)}$$

where $\mathrm{Ind}_C(\cdot, \cdot)$ *is the topological index for* k*-set contractions in the space* $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, *and* $m_-(A)$ *denotes the sum of the multiplicities of the negative eigenvalues of the operator* A.

Chapter 3

Compactness properties

In this chapter we study properties of the Nemytskii operator

$$F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

associated with the function

$$f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}.$$

In particular, we show that F is well-defined, continuous, and satisfies the Lipschitz condition. We also discuss properties of sequences of functions f_n, $n \geq 1$, and the corresponding Nemytskii operators F_n, $n \geq 1$.

Next, we examine the compactness properties of the set

$$\bigcup_{n=1}^{\infty} F_n(D \times U) \tag{3.1}$$

in the space $L^2(\mathbb{R}^N)$, where $D \subset [0, +\infty)$ and $U \subset H^1(\mathbb{R}^N)$ are bounded. We prove that the set (3.1) is relatively compact, provided that the functions f_n, $n \geq 1$, satisfy condition (f2)$'$ (see p. ix) with a common function $m \in L^2(\mathbb{R}^N)$. If this condition is not assumed, we show that the Hausdorff measure of non-compactness of the set (3.1) can be estimated by $\widehat{\varrho}(l_\infty)\beta_{L^2}(U)$. Here, the number $\widehat{\varrho}(l_\infty) \geq 0$ is defined by (14), and the function l_∞ is the $L^\infty(\mathbb{R}^N)$-part of a Kato-Rellich type function arising from condition (f1).

Subsequently, we establish that the translation along trajectories operator associated with the nonlinear damped wave equation (1) is a well-defined, continuous k-set contraction. This contraction is with respect to $\chi_{s,V}$, the Hausdorff measure of non-compactness on the space $(\mathbb{X}, \|\cdot\|_{s,V})$, where $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and the norm $\|\cdot\|_{s,V}$ is given by (2.60).

The chapter concludes by considering a sequence of frequency changing evolution equations with εT-periodic mild solutions, as the parameter $\varepsilon > 0$ tends to 0. We then prove that the corresponding sequence of periodic points is relatively compact.

3.1 Nemytskii operator

Suppose that $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions (f1) and (f2) (see condition (C) on p. viii, condition (f1) on p. ix, and condition (f2) on p. ix, respectively). We then define the *Nemytskii operator* $F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$[F(t, u)](x) = f(t, x, u(x)) \tag{3.2}$$

for all $u \in H^1(\mathbb{R}^N)$, all $t \geq 0$, and almost every $x \in \mathbb{R}^N$.

From Lemma 2.1.3, we deduce basic properties of the linear operators L_∞, L_0, and L, which are defined as multiplication operators by l_∞, l_0, and l, respectively, where $l = l_\infty + l_0$ is a Kato-Rellich type function.

Corollary 3.1.1. *Let l be a non-negative Kato-Rellich type function, i.e., l satisfies* (8a), (8b), *and* (9). *Consider the linear operators L_∞, L_0, and L:*

$$L_\infty : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad L_0 : \mathcal{D}(L_0) \to L^2(\mathbb{R}^N), \quad L : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \tag{3.3a}$$

defined by

$$[L_\infty u](x) = l_\infty(x)u(x), \quad [L_0 u](x) = l_0(x)u(x), \quad [Lu](x) = l(x)u(x), \tag{3.3b}$$

for almost every $x \in \mathbb{R}^N$, where $\mathcal{D}(L_0) = \mathcal{D}(\boldsymbol{V}_0)$ (see (2.8)*).*

Then the linear operators L_∞, L_0 are well-defined and bounded. Furthermore, the space $H^1(\mathbb{R}^N)$ is continuously embedded into $\mathcal{D}(L_0)$, and the linear operator $L = L_\infty + L_0$ is well-defined and bounded. In addition, for all $u \in H^1(\mathbb{R}^N)$,

$$\|Lu\|_{L^2} \le \|l_\infty\|_{L^\infty}\|u\|_{L^2} + C\|l_0\|_{L^p}\|u\|_{H^1}$$

where $C = C(N,p) > 0$ is a constant.

Proposition 3.1.2. *Suppose that $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying* (f1) *and* (f2)*.*

(i) The Nemytskii operator F given by the formula (3.2)*, is well-defined, i.e., $F(t,u)$ belongs to the space $L^2(\mathbb{R}^N)$, for any $t \in [0,+\infty)$ and $u \in H^1(\mathbb{R}^N)$. Moreover, F is continuous and satisfies the following conditions:*

(**F1**) *F satisfies the Lipschitz condition, i.e., there exists $C > 0$ such that, for all $t \ge 0$ and u_1, $u_2 \in H^1(\mathbb{R}^N)$,*

$$\|F(t,u_1) - F(t,u_2)\|_{L^2} \le C\|u_1 - u_2\|_{H^1},$$

with a constant $C = \|L\|_{\mathcal{L}(H^1,L^2)}$, where the linear operator L is given by (3.3a) *and* (3.3b)*,*

(**F2**) *F is bounded at zero, i.e., there exists $M_0 > 0$ such that*

$$\|F(t,0)\|_{L^2} \le M_0 \quad \textit{for all } t \ge 0,$$

with a constant $M_0 = \|m_0\|_{L^2}$, where function $m_0 \in L^2(\mathbb{R}^N)$ is from condition (f2)*.*

(ii) Assume additionally that f is T-periodic in time (cf. condition (P) on p. ix). Then the function $\widehat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, given by

$$\widehat{f}(x,u) = \frac{1}{T}\int_0^T f(t,x,u)\,dt, \tag{3.4}$$

for almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$, is also a Carathéodory function satisfying (f1) *and* (f2) *with the same functions m_0 and l. Moreover, $\widehat{F} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, the Nemytskii operator associated with $\widehat{f}$, is well-defined, continuous, satisfies condition* (**F1**)*, and*

$$\widehat{F}(u) = \frac{1}{T}\int_0^T F(t,u)\,dt \quad \textit{for all } u \in H^1(\mathbb{R}^N). \tag{3.5}$$

Proof. (i) First, in view of [53, Thm. 10.58], for any $t \in [0,+\infty)$ and $u \in H^1(\mathbb{R}^N)$, the function $F(t,u) : \mathbb{R}^N \to \mathbb{R}$ is measurable. We set $t \in [0,+\infty)$ and $u \in H^1(\mathbb{R}^N)$. From conditions (f1) and (f2) we deduce that, for almost every $x \in \mathbb{R}^N$,

$$|f(t,x,u(x))| \le l(x)\,|u(x)| + m_0(x) \tag{3.6}$$

and, by Corollary 3.1.1, $L|u| + m_0 \in L^2(\mathbb{R}^N)$, thus $F(t,u) \in L^2(\mathbb{R}^N)$. Condition (**F2**) is a direct consequence of (f2). To prove (**F1**) we use condition (f1) and the boundedness of $L : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ (see Corollary 3.1.1):

$$\|F(t,u_1) - F(t,u_2)\|_{L^2} \le \|L(u_1 - u_2)\|_{L^2} \le \|L\|_{\mathcal{L}(H^1,L^2)} \|u_1 - u_2\|_{H^1}.$$

Now, let $t_n \to t_0$ in $[0,+\infty)$ and $u_n \to u_0$ in $H^1(\mathbb{R}^N)$. Due to the Lipschitz condition one gets

$$\begin{aligned} \|F(t_n,u_n) - F(t_0,u_0)\|_{L^2} &\le \|F(t_n,u_n) - F(t_n,u_0)\|_{L^2} + \|F(t_n,u_0) - F(t_0,u_0)\|_{L^2} \\ &\le \|L\|_{\mathcal{L}(H^1,L^2)} \|u_n - u_0\|_{H^1} + \|F(t_n,u_0) - F(t_0,u_0)\|_{L^2}. \end{aligned} \tag{3.7}$$

As f is a Carathéodory function, $f(t_n,x,u_0(x)) \xrightarrow{n\to\infty} f(t_0,x,u_0(x))$, for almost every $x \in \mathbb{R}^N$. Furthermore, by virtue of (f1) and (f2), one has, for almost every $x \in \mathbb{R}^N$ and all $n \ge 1$,

$$|f(t_n,x,u_0(x))| \le l(x)\,|u_0(x)| + m_0(x).$$

Since $L|u_0| + m_0 \in L^2(\mathbb{R}^N)$, by the Lebesgue Dominated Convergence Theorem we get

$$F(t_n,u_0) \xrightarrow{n\to\infty} F(t_0,u_0) \quad \text{in } L^2(\mathbb{R}^N).$$

This together with (3.7) yields

$$F(t_n,u_n) \xrightarrow{n\to\infty} F(t_0,u_0) \quad \text{in } L^2(\mathbb{R}^N),$$

which proves that $F : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous.

(ii) Observe that, for almost every $x \in \mathbb{R}^N$ and all $u_1,\ u_2 \in \mathbb{R}$,

$$|\widehat{f}(x,u_1) - \widehat{f}(x,u_2)| \le \frac{1}{T}\int_0^T |f(t,x,u_1) - f(t,x,u_2)|\,dt \le l(x)|u_1 - u_2|$$

that is, $\widehat{f}$ fulfills (f1) with function l. Let $u \in \mathbb{R}$ and approximate $\widehat{f}(\cdot,u) : \mathbb{R}^N \to \mathbb{R}$ by the Riemann sums:

$$\widehat{f}(x,u) = \frac{1}{T}\lim_{n\to\infty}\sum_{k=1}^{n} f(kT/n,x,u)(T/n) = \frac{1}{T}\lim_{n\to\infty} g_n(x) \quad \text{for almost every } x \in \mathbb{R}^N$$

where $g_n : \mathbb{R}^N \to \mathbb{R}$, $n \ge 1$, are given by

$$g_n(x) = \sum_{k=1}^{n} f(kT/n,x,u)(T/n).$$

Since f is a Carathéodory function, g_n, $n \ge 1$, are measurable, thus, function $\widehat{f}(\cdot,u) : \mathbb{R}^N \to \mathbb{R}$ is measurable. Note that, condition (f1) for $\widehat{f}$ implies that $\widehat{f}(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \mathbb{R}^N$. Hence, $\widehat{f}$ satisfies the Carathéodory condition. In view of condition (f2) for f, we have, for almost every $x \in \mathbb{R}^N$,

$$|\widehat{f}(x,0)| \le \frac{1}{T}\int_0^T |f(t,x,0)|\,dt \le m_0(x),$$

which shows that $\widehat{f}$ satisfies (f2) with m_0. Since $\widehat{f}$ is a Carathéodory function satisfying the Lipschitz condition with a Kato-Rellich type function l, based on (i), we claim that the Nemytskii operator $\widehat{F}$ is well-defined, continuous and fulfills the Lipschitz condition (**F1**).

Now, we show (3.5). Let $u \in H^1(\mathbb{R}^N)$ and denote

$$G = \frac{1}{T}\int_0^T F(t,u)\,dt.$$

Consider the Riemann sums for G:

$$G_n = \frac{1}{T}\sum_{k=1}^{n} F(kT/n, u)(T/n).$$

Since $G_n \to G$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, we may assume, up to a subsequence, that $G_n(x) \to G(x)$ as $n \to \infty$, for almost every $x \in \mathbb{R}^N$ – see Proposition A.2.2 (ii). On the other hand, for almost every $x \in \mathbb{R}^N$,

$$\begin{aligned} G_n(x) &= \frac{1}{T}\sum_{k=1}^{n} F(kT/n, u)(x)(T/n) \\ &= \frac{1}{T}\sum_{k=1}^{n} f(kT/n, x, u(x))(T/n) \xrightarrow{n\to\infty} \frac{1}{T}\int_0^T f(t, x, u(x))\, dt = \widehat{f}(x, u(x)) \end{aligned}$$

thus, $G(x) = \widehat{f}(x, u(x))$ for almost every $x \in \mathbb{R}^N$, that is, $G = \widehat{F}(u)$, as desired. □

Consider sequence of Carathéodory functions

$$f_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad n \geq 1,$$

satisfying (f1) and (f2) with l and m_0 independent of n, and a Carathéodory function

$$f_0 : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$$

satisfying (f1) and (f2). We say that $f_n \to f_0$ in the sense of condition (f0) if

(f0) $f_n(t, x, u) \xrightarrow{n\to\infty} f_0(t, x, u)$ for almost every $x \in \mathbb{R}^N$, all $u \in \mathbb{R}$, and uniformly on bounded subsets of $[0, +\infty)$ with respect to t.

By F_n, $n \geq 1$, we denote the Nemytskii operators associated with f_n, $n \geq 1$, and by F_0 the Nemytskii operator associated with f_0. Observe, in view of Proposition 3.1.2 (i), that operators F_n, $n \geq 1$, and F_0 are well-defined, continuous and fulfill conditions (**F1**), (**F2**).

Proposition 3.1.3. *Let f_n, $n \geq 0$, be Carathéodory functions satisfying conditions* (f1) *and* (f2) *with common functions l and m_0. If $f_n \to f_0$ in the sense of condition* (f0)*, then*

(i) For almost every $x \in \mathbb{R}^N$ and any sequences $t_n \xrightarrow{n\to\infty} t_0$ in $[0, +\infty)$ and $u_n \xrightarrow{n\to\infty} u_0$ in $\mathbb{R}$

$$f_n(t_n, x, u_n) \xrightarrow{n\to\infty} f_0(t_0, x, u_0).$$

(ii) For any sequences $t_n \xrightarrow{n\to\infty} t_0$ in $[0, +\infty)$ and $u_n \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^N)$

$$F_n(t_n, u_n) \xrightarrow{n\to\infty} F_0(t_0, u_0) \quad \text{in } L^2(\mathbb{R}^N). \tag{3.8}$$

In addition, the set

$$\{F_n(t, u) : t \in D, n \geq 1\} \tag{3.9}$$

is relatively compact in $L^2(\mathbb{R}^N)$, for any bounded $D \subset [0, +\infty)$ and $u \in H^1(\mathbb{R}^N)$.

(iii) Let $\mathbb{F}_n : [0, +\infty) \times \mathbb{X} \to \mathbb{X}$, $n \geq 0$, where $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, are given by

$$\mathbb{F}_n(t, u, v) = \big(0, F_n(t, u)\big) \quad \text{for } t \geq 0 \text{ and } (u, v) \in \mathbb{X}.$$

Then the mappings $\mathbb{F}_n$, $n \geq 0$, are continuous and satisfy the Lipschitz condition (F1) *and are bounded at zero* (F2) *(see p. 4) with constants independent of n,*

$$\mathbb{F}_n(t_n, u_n, v_n) \xrightarrow{n\to\infty} \mathbb{F}_0(t_0, u_0, v_0) \ \text{ in } \mathbb{X},$$

for any sequences $t_n \xrightarrow{n\to\infty} t_0$ *in* $[0,+\infty)$ *and* $(u_n, v_n) \xrightarrow{n\to\infty} (u_0, v_0)$ *in* $\mathbb{X}$, *and*

$$\int_0^t \mathbb{F}_n(\tau, u, v)\, d\tau \xrightarrow{n\to\infty} \int_0^t \mathbb{F}_0(\tau, u, v)\, d\tau \quad \text{in } \mathbb{X}, \tag{3.10}$$

for all $t > 0$ *and* $(u, v) \in \mathbb{X}$, *and the set*

$$\{\mathbb{F}_n(t, u, v) : t \in D, n \geq 1\}$$

is relatively compact in $\mathbb{X}$, *for any bounded* $D \subset [0,+\infty)$ *and* $(u, v) \in \mathbb{X}$.

Proof. (i) Due to assumption, function $f_0(\cdot, x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^N$. Hence([1]), in view of the assumed condition (f0),

$$f_n(t_n, x, u) \xrightarrow{n\to\infty} f_0(t_0, x, u)$$

for almost every $x \in \mathbb{R}^N$, all $u \in \mathbb{R}$ and any sequence $t_n \xrightarrow{n\to\infty} t_0$ in $[0,+\infty)$. Moreover, by (f1), for any $n \geq 1$, there exists a set $B_n \subset \mathbb{R}^N$ of measure zero such that, for all $x \in \mathbb{R}^N \setminus B_n$, all $t \geq 0$ and all u_1, $u_2 \in \mathbb{R}$

$$|f_n(t, x, u_1) - f_n(t, x, u_2)| \leq l(x)|u_1 - u_2|. \tag{3.11}$$

Then $B = \bigcup_{n=1}^{\infty} B_n$ has a measure zero and therefore (3.11) holds for almost every $x \in \mathbb{R}^N$, all $n \geq 1$, all $t \geq 0$ and all u_1, $u_2 \in \mathbb{R}$. Consequently, for almost every $x \in \mathbb{R}^N$, any sequences $t_n \xrightarrow{n\to\infty} t_0$ in $[0,+\infty)$ and $u_n \xrightarrow{n\to\infty} u_0$ in $\mathbb{R}$, we have

$$\begin{aligned} |f_n(t_n, x, u_n) - f_0(t_0, x, u_0)| &\leq |f_n(t_n, x, u_n) - f_n(t_n, x, u_0)| + |f_n(t_n, x, u_0) - f_0(t_0, x, u_0)| \\ &\leq l(x)|u_n - u_0| + |f_n(t_n, x, u_0) - f_0(t_0, x, u_0)| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

(ii) Observe that, for all $n \geq 1$,

$$\|F_n(t_n, u_n) - F_0(t_0, u_0)\|_{L^2} \leq \|F_n(t_n, u_n) - F_n(t_n, u_0)\|_{L^2} + \|F_n(t_n, u_0) - F_0(t_0, u_0)\|_{L^2}. \tag{3.12}$$

By condition (f1) we have

$$\|F_n(t_n, u_n) - F_n(t_n, u_0)\|_{L^2} \leq \|Lu_n - Lu_0\|_{L^2} \quad \text{for all } n \geq 1.$$

By Corollary 3.1.1 linear operator L is bounded. Hence, because by the assumption $u_n \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^N)$, we obtain

$$Lu_n \xrightarrow{n\to\infty} Lu_0 \quad \text{in } L^2(\mathbb{R}^N).$$

Consequently,

$$\|F_n(t_n, u_n) - F_n(t_n, u_0)\|_{L^2} \xrightarrow{n\to\infty} 0. \tag{3.13}$$

By the assumption, $f_n \to f_0$ in the sense of condition (f0) and $t_n \xrightarrow{n\to\infty} t_0$ in $[0,+\infty)$. Therefore, by virtue of the point (i),

$$f_n(t_n, x, u_0(x)) \xrightarrow{n\to\infty} f_0(t_0, x, u_0(x)) \quad \text{for almost every } x \in \mathbb{R}^N.$$

On the other hand, by conditions (f1) and (f2),

$$|f_n(t_n, x, u_0(x))| \leq l(x)u_0(x) + m_0(x) \quad \text{for almost every } x \in \mathbb{R}^N \text{ and all } n \geq 1.$$

Moreover, by Corollary 3.1.1, $Lu_0 + m_0 \in L^2(\mathbb{R}^N)$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\|F_n(t_n, u_0) - F_0(t_0, u_0)\|_{L^2} \xrightarrow{n\to\infty} 0. \tag{3.14}$$

([1])Here we apply the following fact: Let (X, d), (Y, p) be metric spaces and functions f, f_n, $n \geq 1$, from X to Y. Then $f_n(x_n) \xrightarrow{n\to\infty} f(x)$ for any sequence $(x_n)_{n\geq 1}$ in X satisfying $x_n \xrightarrow{n\to\infty} x$ if and only if f is continuous and $f_n \xrightarrow{n\to\infty} f$ uniformly on relatively compact subsets of X, cf. [32, Prop. 1.3 (2)].

We apply (3.13) and (3.14) to (3.12) obtaining (3.8).

Let $(w_j)_{j\geq 1}$ be a sequence in the set (3.9). We shall prove that the set $\{w_j\}_{j\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Then, either there exists $n \geq 1$ such that

$$\{w_j\}_{j\geq 1} \cap F_n(D \times \{u\}) \tag{3.15}$$

is an infinite set, or, for any $n \geq 1$, the intersection (3.15) is finite.

We consider the first case. Therefore, we can choose a subsequence $(w_{j_k})_{k\geq 1}$ such that

$$w_{j_k} \in F_n(D \times \{u\}) \quad \text{for all } k \geq 1 \text{ and some } n \geq 1.$$

Without loss of generality we can assume that

$$w_j \in F_n(D \times \{u\}) \quad \text{for all } j \geq 1.$$

Since, by the assumption, the set $D \subset [0, +\infty)$ is bounded, we see that it is relatively compact, and the set $D \times \{u\}$ is relatively compact in $[0, +\infty) \times H^1(\mathbb{R}^N)$. By Proposition 3.1.2 (i) the map F_n is continuous. Hence, the set $F_n(D \times \{u\})$ is relatively compact in $L^2(\mathbb{R}^N)$. As $\{w_j\}_{j\geq 1} \subset F_n(D \times \{u\})$, we obtain that the set $\{w_j\}_{j\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$.

Let us consider the second case, that is, the intersection (3.15) is finite, for any $n \geq 1$. As the set $\{w_j\}_{j\geq 1}$ consists of infinitely many elements, we can choose subsequences $(w_{j_k})_{k\geq 1}$ and $(F_{n_k})_{k\geq 1}$ such that

$$w_{j_k} \in F_{n_k}(D \times \{u\}) \quad \text{for all } k \geq 1.$$

Without loss of generality we can assume that

$$w_j \in F_j(D \times \{u\}) \quad \text{for all } j \geq 1.$$

Hence, we can choose a sequence $(t_j)_{j\geq 1}$ such that

$$w_j = F_j(t_j, u) \quad \text{for all } j \geq 1.$$

Since the set D is bounded, we can assume, up to a subsequence, that $t_j \xrightarrow{j\to\infty} t_0$ in $\overline{D}$. Thus, applying point (ii), we get that

$$w_j \xrightarrow{j\to\infty} F_0(t_0, u) \quad \text{in } L^2(\mathbb{R}^N).$$

Therefore, we proved that the set $\{w_j\}_{j\geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$.

(iii) Let us define $j : L^2(\mathbb{R}^N) \to \mathbb{X}$, $j(v) = (0, v)$, and $\Pi : \mathbb{X} \to H^1(\mathbb{R}^N)$, $\Pi(u, v) = u$. We see that j and Π are bounded linear operators and

$$\mathbb{F}_n(t, u, v) = j\Big(F_n\big(t, \Pi(u, v)\big)\Big), \tag{3.16}$$

for any $n \geq 0$, $t \geq 0$ and $(u, v) \in \mathbb{X}$. By Proposition 3.1.2 (i) the mappings F_n, $n \geq 0$, are continuous and satisfy (**F1**) and (**F2**) with constants independent of n. Taking this fact together with (3.16) we get that $\mathbb{F}_n$, $n \geq 0$, are continuous and satisfy (F1) and (F2) with common constants.

Let $t_n \xrightarrow{n\to\infty} t_0$ in $[0, +\infty)$ and $(u_n, v_n) \xrightarrow{n\to\infty} (u_0, v_0)$ in $\mathbb{X}$. As Π is a bounded linear operator, we have $\Pi(u_n, v_n) \to \Pi(u_0, v_0)$ in $H^1(\mathbb{R}^N)$. Hence, by the assertion (ii)

$$F_n\big(t_n, \Pi(u_n, v_n)\big) \xrightarrow{n\to\infty} F_0\big(t_0, \Pi(u_0, v_0)\big) \quad \text{in } L^2(\mathbb{R}^N).$$

Since j is a bounded linear operator, we get that

$$\mathbb{F}_n(t_n, u_n, v_n) = j\Big(F_n\big(t_n, \Pi(u_n, v_n)\big)\Big) \xrightarrow{n\to\infty} j\Big(F_0\big(t_0, \Pi(u_0, v_0)\big)\Big) = \mathbb{F}_0(t_0, u_0, v_0). \tag{3.17}$$

We take some $t > 0$ and $(u, v) \in \mathbb{X}$. Since $\mathbb{F}_n$, $n \geq 0$, satisfy conditions (F1) and (F2) with constants independent of n, there exists a constant $C > 0$ such that, for all $n \geq 0$, all $t \in [0, +\infty)$ and all $(u, v) \in \mathbb{X}$,

$$\|\mathbb{F}_n(t, u, v)\|_{\mathbb{X}} \leq C(1 + \|(u, v)\|_{\mathbb{X}}).$$

Taking the above estimate together with (3.17) and using the Dominated Convergence Theorem we get (3.10).

Now, let $D \subset [0, +\infty)$ be bounded and $(u, v) \in \mathbb{X}$. Observe, again by the assertion (ii), that the set

$$\{F_n(t, \Pi(u, v)) : t \in D, n \geq 1\}$$

is relatively compact in $L^2(\mathbb{R}^N)$. Moreover, we have

$$\begin{aligned} \{\mathbb{F}_n(t, u, v) : t \in D, n \geq 1\} &= \left\{ j\Big(F_n\big(t, \Pi(u, v)\big)\Big) : t \in D, n \geq 1 \right\} \\ &= j\left(\left\{F_n\big(t, \Pi(u, v)\big) : t \in D, n \geq 1\right\}\right). \end{aligned}$$

As j is a bounded linear operator, we see that the set

$$\{\mathbb{F}_n(t, u, v) : t \in D, n \geq 1\}$$

is relatively compact in $\mathbb{X}$. □

Remark 3.1.4. Observe that we may slightly weaken the assumptions on the function f_0 in Proposition 3.1.3. Namely, let f_n, $n \geq 1$, be Carathéodory functions satisfying conditions (f1) and (f2) with common functions l and m_0. If the function f_0 is a Carathéodory function such that $f_n \to f_0$ in the sense of condition (f0), then f_0 satisfies conditions (f1) and (f2) with the same functions l and m_0.

In fact, from the proof of Proposition 3.1.2 (i), we deduce that inequality (3.11) holds for almost every $x \in \mathbb{R}^N$, all $n \geq 1$, all $t \geq 0$, and all u_1, $u_2 \in \mathbb{R}$. Passing to the limit as $n \to \infty$ in (3.11), we obtain that f_0 satisfies condition (f1) with the function l. Similarly, we observe that, for almost every $x \in \mathbb{R}$, all $n \geq 1$, and all $t \geq 0$,

$$|f_n(t, x, 0)| \leq m_0(x).$$

Hence, after passing to the limit as $n \to \infty$, we conclude that f_0 satisfies condition (f2) with the function m_0. □

We also consider a parametrized version of the above results. Let (M, d) be a compact metric space and a function

$$h : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times M \to \mathbb{R}$$

be such that

($\widetilde{C}$) (*Carathéodory condition*) function

$$(t, u, \mu) \mapsto h(t, x, u, \mu)$$

is continuous on $[0, +\infty) \times \mathbb{R} \times M$, for almost every $x \in \mathbb{R}^N$, and function

$$x \mapsto h(t, x, u, \mu)$$

is measurable, for all $t \geq 0$, all $u \in \mathbb{R}$, and all $\mu \in M$;

(h1) (*Lipschitz condition*) there exists a non-negative Kato-Rellich type function $l : \mathbb{R}^N \to \mathbb{R}$, i.e., l fulfilling (8a), (8b), and (9), such that

$$|h(t,x,u_1,\mu) - h(t,x,u_2,\mu)| \le l(x)|u_1 - u_2|,$$

for almost every $x \in \mathbb{R}^N$, all $t \ge 0$, u_1, $u_2 \in \mathbb{R}$, and $\mu \in M$;

(h2) (*Boundedness at zero*) there exists $m_0 \in L^2(\mathbb{R}^N)$ such that

$$|h(t,x,0,\mu)| \le m_0(x),$$

for almost every $x \in \mathbb{R}^N$, all $t \ge 0$, $\mu \in M$.

We define the *Nemytskii operator*

$$H : [0,+\infty) \times H^1(\mathbb{R}^N) \times M \to L^2(\mathbb{R}^N)$$

associated with function h by

$$[H(t,u,\mu)](x) = h(t,x,u(x),\mu), \tag{3.18}$$

for all $u \in H^1(\mathbb{R}^N)$, almost every $x \in \mathbb{R}^N$, all $t \ge 0$, $u \in \mathbb{R}$, and $\mu \in M$.

Corollary 3.1.5. *Suppose that a Carathéodory function h satisfies* (h1) *and* (h2). *Then the Nemytskii operator H given by the formula* (3.18) *is well-defined, i.e., $H(t,u,\mu)$ belongs to the space $L^2(\mathbb{R}^N)$, for any $t \ge 0$, $u \in H^1(\mathbb{R}^N)$, and $\mu \in M$. Moreover, H is continuous and satisfies the following conditions:*

(**H1**) *H satisfies Lipschitz condition, i.e., there exists $C > 0$ such that*

$$\|H(t,u_1,\mu) - H(t,u_2,\mu)\|_{L^2} \le C\|u_1 - u_2\|_{H^1},$$

for all $t \ge 0$, u_1, $u_2 \in H^1(\mathbb{R}^N)$, and $\mu \in M$;

(**H2**) *H is bounded at zero, i.e., there exists $M_0 > 0$ such that*

$$\|H(t,0,\mu)\|_{L^2} \le M_0,$$

for all $t \ge 0$ and $\mu \in M$.

Proof. By the Carathéodory condition and [53, Thm. 10.58], function $H(t,u,\mu) : \mathbb{R}^N \to \mathbb{R}$ is measurable for any $t \ge 0$, $u \in H^1(\mathbb{R}^N)$, and $\mu \in M$. Similarly as in Proposition 3.1.2 (i), we obtain that $H(t,u,\mu)$ belongs to the space $L^2(\mathbb{R}^N)$, for any $t \ge 0$, $u \in H^1(\mathbb{R}^N)$, and $\mu \in M$, using (h1), (h2) and the fact that $L|u| + m_0 \in L^2(\mathbb{R}^N)$ for any $u \in H^1(\mathbb{R}^N)$ (see Corollary 3.1.1).

As in Proposition 3.1.2 (i), from (h1) and the boundedness of the linear operator L (see Corollary 3.1.1), we deduce (**H1**). We see also that (**H2**) is a direct consequence of (h2).

Now, let $t_n \xrightarrow{n\to\infty} t_0$ in $[0,+\infty)$, $u_n \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^N)$, and $\mu_n \xrightarrow{n\to\infty} \mu_0$ in M. We define sequence of functions

$$f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad n \ge 1,$$

by the formula

$$f_n(t,x,u) = h(t,x,u,\mu_n),$$

and function

$$f_0 : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$$

by the formula

$$f_0(t,x,u) = h(t,x,u,\mu_0).$$

Then f_n, $n \geq 1$, are Carathéodory functions fulfilling (f1) and (f2) with common functions l and m_0. Furthermore, f_0 is also a Carathéodory function satisfying (f1) and (f2). We see that

$$H(t, u, \mu_n) = F_n(t, u) \quad \text{for all } n \geq 1,\ t \geq 0,\ u \in H^1(\mathbb{R}^N),$$

and

$$H(t, u, \mu_0) = F_0(t, u) \quad \text{for all } t \geq 0,\ u \in H^1(\mathbb{R}^N).$$

Carathéodory condition implies that function

$$(t, u, \mu) \mapsto h(t, x, u, \mu)$$

is continuous on $[0, +\infty) \times \mathbb{R} \times M$ for almost every $x \in \mathbb{R}^N$. Hence, in particular,

$$h(t, x, u, \mu_n) \xrightarrow{n \to \infty} h(t, x, u, \mu_0),$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, all $u \in \mathbb{R}$, and all $\mu \in M$. In addition, this convergence is uniform on bounded subsets of $[0, +\infty)$ with respect to t. Therefore, condition (f0) is satisfied. Thus, in the light of Proposition 3.1.3 (ii),

$$H(t_n, u_n, \mu_n) = F_n(t_n, u_n) \xrightarrow{n \to \infty} F_0(t_0, u_0) = H(t_0, u_0, \mu_0) \quad \text{in } L^2(\mathbb{R}^N),$$

which shows that H is continuous, and ends the proof. □

3.2 Compactness properties for L^2-bounded nonlinearity

In this section we consider functions $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying condition:

(f2)′ there exists $m \in L^2(\mathbb{R}^N)$ such that

$$|f(t, x, u)| \leq m(x),$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, $u \in \mathbb{R}$.

As before, by

$$F_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad n \geq 1,$$

we denote the Nemytskii operators associated with functions

$$f_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad n \geq 1.$$

Now, we show that under assumption (f2)′ the set (3.1) is relatively compact in $L^2(\mathbb{R}^N)$.

Proposition 3.2.1. *Let f_n, $n \geq 1$, be a sequence of Carathéodory functions satisfying conditions* (f1) *and* (f2)′ *with common functions l and m. Assume that f_0 is a Carathéodory function satisfying* (f1), (f2)′, *and such that $f_n \to f_0$ in the sense of condition* (f0) *(see p. 50).*

If sets $D \subset [0, +\infty)$ and $U \subset H^1(\mathbb{R}^N)$ are bounded, then the set

$$\bigcup_{n=1}^{\infty} F_n(D \times U)$$

is relatively compact in $L^2(\mathbb{R}^N)$.

Proof. Let $(w_j)_{j\geq 1}$ be a sequence in $\bigcup_{n=1}^{\infty} F_n(D\times U)$. Then, either there exists $n\geq 1$ such that

$$\{w_j\}_{j\geq 1}\cap F_n(D\times U) \tag{3.19}$$

is an infinite set or, for any $n\geq 1$, the intersection (3.19) is finite.

We consider the first case. Therefore, there exists a subsequence $(w_{j_k})_{k\geq 1}$ such that

$$w_{j_k}\in F_n(D\times U) \quad \text{for all } k\geq 1,$$

and some $n\geq 1$. Without loss of generality we may assume that

$$w_j\in F_n(D\times U) \quad \text{for all } j\geq 1.$$

Hence, there exist sequences $(t_j)_{j\geq 1}$ in D and $(u_j)_{j\geq 1}$ in U such that

$$w_j=F_n(t_j,u_j) \quad \text{for all } j\geq 1.$$

Since the set $D\subset[0,+\infty)$ is bounded, we may assume, extracting a subsequence if necessary, that $t_j\xrightarrow{j\to\infty} t_0$ in $\overline{D}$. Furthermore, based on the Eberlein-Šmulian Theorem (see [9, Thm. 3.18 and Thm. 3.19]), we may assume, passing possibly to a subsequence, that

$$u_j \xrightharpoonup{j\to\infty} u_0 \quad \text{in } H^1(\mathbb{R}^N). \tag{3.20}$$

We put

$$w_0=F_n(t_0,u_0).$$

Let $m\in L^2(\mathbb{R}^N)$ be as in condition (f2)$'$ and fix some $\varepsilon>0$. We take $r>0$ such that

$$\|m\|_{L^2(\mathbb{R}^N\setminus B(0,r))}<\varepsilon. \tag{3.21}$$

Then

$$\{w_j\}_{j\geq 1}\subset\{w_j\mathbf{1}_{B(0,r)}\}_{j\geq 1}+\{w_j\mathbf{1}_{\mathbb{R}^N\setminus B(0,r)}\}_{j\geq 1}. \tag{3.22}$$

Note, by condition (f2)$'$ and (3.21), that

$$\|w_j\mathbf{1}_{\mathbb{R}^N\setminus B(0,r)}\|_{L^2}=\|w_j\|_{L^2(\mathbb{R}^N\setminus B(0,r))}\leq\|m\|_{L^2(\mathbb{R}^N\setminus B(0,r))}<\varepsilon \quad \text{for all } j\geq 1. \tag{3.23}$$

In view of the Rellich-Kondrachov Theorem (see Theorem A.2.5), possibly passing to a subsequence, we get that

$$u_j\xrightarrow{j\to\infty} v \quad \text{in } L^2(B(0,r)),$$

for some $v\in L^2(B(0,r))$. In particular,

$$u_j\rightharpoonup v \quad \text{in } L^2(B(0,r)).$$

On the other hand, from (3.20) we deduce that

$$u_j \xrightharpoonup{j\to\infty} u_0 \quad \text{in } L^2(\mathbb{R}^N),$$

and consequently

$$u_j \xrightharpoonup{j\to\infty} u_0 \quad \text{in } L^2(B(0,r)).$$

Uniqueness of a weak limit yields $u_0(x)=v(x)$ for almost every $x\in B(0,r)$, hence

$$u_j\mathbf{1}_{B(0,r)}\xrightarrow{j\to\infty} u_0\mathbf{1}_{B(0,r)} \ \text{ in } L^2(\mathbb{R}^N). \tag{3.24}$$

Therefore, passing to a subsequence if necessary,

$$u_j(x)\mathbf{1}_{B(0,r)}(x)\xrightarrow{j\to\infty} u_0(x)\mathbf{1}_{B(0,r)}(x) \quad \text{for almost every } x\in\mathbb{R}^N; \tag{3.25}$$

see Proposition A.2.2 (ii). Observe that

$$w_j(x)\mathbf{1}_{B(0,r)}(x) = f_n\big(t_j, x, u_j(x)\mathbf{1}_{B(0,r)}(x)\big)\mathbf{1}_{B(0,r)}(x),$$

for all $j \geq 0$ and almost every $x \in \mathbb{R}^N$. Since f_n is a Carathéodory function, from (3.25) we deduce that

$$w_j(x)\mathbf{1}_{B(0,r)}(x) \xrightarrow{j\to\infty} w_0(x)\mathbf{1}_{B(0,r)}(x) \text{ for almost every } x \in \mathbb{R}^N.$$

This together with condition (f2)$'$ allow us to apply the Lebesgue Dominated Convergence Theorem, thus,

$$w_j\mathbf{1}_{B(0,r)} \xrightarrow{j\to\infty} w_0\mathbf{1}_{B(0,r)} \quad \text{in } L^2(\mathbb{R}^N).$$

This means that the set $\{w_j\mathbf{1}_{B(0,r)}\}_{j\geq1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Since $\varepsilon > 0$ is arbitrary, combining this with (3.22) and (3.23) we obtain that $\{w_j\}_{j\geq1}$ is relatively compact in $L^2(\mathbb{R}^N)$.

We consider now the second case, i.e., for any $n \geq 1$, the intersection (3.19) is finite. Since sequence $(w_j)_{j\geq1}$ consists of infinitely many elements, we get the existence of subsequences $(w_{j_k})_{k\geq1}$ and $(F_{n_k})_{k\geq1}$ such that

$$w_{j_k} \in F_{n_k}(D \times U) \quad \text{for all } k \geq 1.$$

As $j_k \xrightarrow{k\to\infty} \infty$ and $n_k \xrightarrow{k\to\infty} \infty$, we may assume that

$$w_j \in F_j(D \times U) \quad \text{for all } j \geq 1.$$

Hence, there exist sequences $(t_j)_{j\geq1}$ in D and $(u_j)_{j\geq1}$ in U such that

$$w_j = F_j(t_j, u_j) \quad \text{for all } j \geq 1.$$

Let $\varepsilon>0$ be arbitrary and $r>0$ be such that (3.21) is satisfied. Then, using condition (f2)$'$,

$$\{w_j\}_{j\geq1} \subset \{w_j\mathbf{1}_{B(0,r)}\}_{j\geq1} + B_{L^2}(0, \varepsilon). \tag{3.26}$$

As before, we may assume, extracting a subsequence if necessary, that $t_j \xrightarrow{j\to\infty} t_0$ in $\overline{D}$,

$$u_j \xrightharpoonup{j\to\infty} u_0 \quad \text{in } H^1(\mathbb{R}^N),$$

and

$$u_j(x)\mathbf{1}_{B(0,r)}(x) \xrightarrow{j\to\infty} u_0(x)\mathbf{1}_{B(0,r)}(x) \quad \text{for almost every } x \in \mathbb{R}^N. \tag{3.27}$$

Let

$$w_0 = F_0(t_0, u_0).$$

Observe that

$$w_j(x)\mathbf{1}_{B(0,r)}(x) = f_j\big(t_j, x, u_j(x)\mathbf{1}_{B(0,r)}(x)\big)\mathbf{1}_{B(0,r)}(x),$$

for all $j \geq 0$ and almost every $x \in \mathbb{R}^N$. Recall that $t_j \xrightarrow{j\to\infty} t_0$ in $\overline{D}$, (3.27) holds, and, by assumption, $f_j \to f_0$ in the sense of condition (f0). Therefore, Proposition 3.1.3 (i) yields

$$w_j(x)\mathbf{1}_{B(0,r)}(x) \xrightarrow{j\to\infty} w_0(x)\mathbf{1}_{B(0,r)}(x) \quad \text{for almost every } x \in \mathbb{R}^N.$$

As all the functions f_j, $j \geq 1$, satisfy (f2)$'$ with a common function $m \in L^2(\mathbb{R}^N)$, we may use the Lebesgue Dominated Convergence Theorem:

$$w_j\mathbf{1}_{B(0,r)} \xrightarrow{j\to\infty} w_0\mathbf{1}_{B(0,r)} \quad \text{in } L^2(\mathbb{R}^N).$$

Therefore, $\{w_j\mathbf{1}_{B(0,r)}\}_{j\geq1}$ is relatively compact in $L^2(\mathbb{R}^N)$. Since $\varepsilon > 0$ is arbitrary, this together with (3.26) show that the set $\{w_j\}_{j\geq1}$ is relatively compact in $L^2(\mathbb{R}^N)$. The proof is completed. □

As before we formulate a parametrized version of the result for sequence of functions. Namely, consider a Carathéodory function $h : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \times M \to \mathbb{R}$ satisfying (h1) and

(h2)′ there exists $m \in L^2(\mathbb{R}^N)$ such that

$$|h(t,x,u,\mu)| \le m(x),$$

for almost every $x \in \mathbb{R}^N$, all $t \ge 0$, $u \in \mathbb{R}$, and $\mu \in M$,

where (M,d) is a compact metric space. By

$$H : [0,+\infty) \times H^1(\mathbb{R}^N) \times M \to L^2(\mathbb{R}^N)$$

we denote the Nemytskii operator associated with h.

Corollary 3.2.2. *Assume that a Carathéodory function h satisfies conditions* (h1) *and* (h2)′. *Then, for any bounded subsets $D \subset [0,+\infty)$ and $U \subset H^1(\mathbb{R}^N)$, the set $H(D \times U \times M)$ is relatively compact in $L^2(\mathbb{R}^N)$.*

Proof. We take a sequence $(w_n)_{n\ge1}$ in $H(D \times U \times M)$. Then there exist sequences $(t_n)_{n\ge1}$ in D, $(u_n)_{n\ge1}$ in $H^1(\mathbb{R}^N)$ and $(\mu_n)_{n\ge1}$ in M such that $w_n = H(t_n,u_n,\mu_n)$ for any $n \ge 1$. We may assume, extracting a subsequence if necessary, that $\mu_n \xrightarrow{n\to\infty} \mu_0$ in M. We define functions

$$f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad n \ge 1,$$

by the formula

$$f_n(t,x,u) = h(t,x,u,\mu_n),$$

and function

$$f_0 : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$$

by

$$f_0(t,x,u) = h(t,x,u,\mu_0).$$

Since h is a Carathéodory function satisfying (h1) and (h2)′, we see that f_n, $n \ge 1$ are also Carathéodory functions satisfying conditions (f1) and (f2)′ with common l and m. Next, in the same way, we get that f_0 satisfies Carathéodory condition, (f1) and (f2)′. By F_n, $n \ge 0$, we denote the Nemytskii operators associated with f_n, $n \ge 0$. Hence, $\{w_n\}_{n\ge1} \subset \bigcup_{n=1}^{\infty} F_n(D \times U)$. Moreover, as in Corollary 3.1.5, from the continuity requirement in Carathéodory condition for h we deduce that $f_n \to f_0$ in the sense of condition (f0). Therefore, by Proposition 3.2.1, the set $\bigcup_{n=1}^{\infty} F_n(D \times U)$ is relatively compact in $L^2(\mathbb{R}^N)$, thus, $\{w_n\}_{n\ge1}$ is also relatively compact in $L^2(\mathbb{R}^N)$. The proof is finished. □

3.3 Compactness properties for nonlinearity with sublinear growth

We begin by the main result of this section providing the estimate of the Hausdorff measure of non-compactness of the set (3.1).

Proposition 3.3.1. *Suppose that Carathéodory functions f_n, $n \ge 1$, satisfy conditions* (f1), (f2) *with common functions l, m_0, respectively. Let f_0 be a Carathéodory function satisfying conditions* (f1), (f2), *and such that $f_n \to f_0$ in the sense of condition* (f0) *(see p. 50).*

If the sets $D \subset [0,+\infty)$ and $U \subset H^1(\mathbb{R}^N)$ are bounded, then

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\Big) \le \widehat{\varrho}(l_\infty)\, \chi_{L^2}(U) \tag{3.28}$$

where the number $\widehat{\varrho}(l_\infty) \ge 0$ is given by (14).

Proof of Proposition 3.3.1 relies on the following lemma concerning compactness on $L^2(B(0,R))$ for some $R>0$.

Lemma 3.3.2. *Let f_n, $n\geq 1$, f_0, $D\subset[0,+\infty)$ and $U\subset H^1(\mathbb{R}^N)$ be as in Proposition 3.3.1. Then, for any $R>0$, the set*

$$\bigcup_{n=1}^{\infty} F_n(D\times U)\mathbf{1}_{B(0,R)}$$

is relatively compact in $L^2(\mathbb{R}^N)$, where

$$W\mathbf{1}_{B(0,R)}=\{u\mathbf{1}_{B(0,R)}\in L^2(\mathbb{R}^N): u\in W\},$$

for any subset $W\subset L^2(\mathbb{R}^N)$.

Proof. Let $(w_m)_{m\geq1}$ be a sequence in $\bigcup_{n=1}^{\infty}F_n(D\times U)\mathbf{1}_{B(0,R)}$. We distinguish two possible cases. In the first case we assume that there exists $n\geq 1$ such that infinitely many elements of the sequence $(w_m)_{m\geq 1}$ belong to $F_n(D\times U)\mathbf{1}_{B(0,R)}$. Hence, there exists a subsequence $(w_{m_k})_{k\geq1}$ such that

$$w_{m_k}\in F_n(D\times U)\mathbf{1}_{B(0,R)} \quad \text{for all } k\geq 1.$$

In the second case we can choose subsequences $(w_{m_k})_{k\geq1}$ and $(F_{n_k})_{k\geq1}$ such that

$$w_{m_k}\in F_{n_k}(D\times U)\mathbf{1}_{B(0,R)} \quad \text{for all } k\geq 1. \tag{3.29}$$

We consider the first case. Without loss of generality we may assume that

$$w_m\in F_n(D\times U)\mathbf{1}_{B(0,R)} \quad \text{for all } m\geq 1.$$

Hence, there exist sequences $(t_m)_{m\geq1}$ in D and $(u_m)_{m\geq1}$ in U such that

$$w_m=F_n(t_m,u_m)\mathbf{1}_{B(0,R)} \quad \text{for all } m\geq 1.$$

Since D is bounded, we can assume, extracting a subsequence if necessary, that $t_m\to t_0$ for some $t_0\in\overline{D}$. By the Eberlein-Šmulian theorem, we can also assume, again passing to a subsequence if necessary, that

$$u_m\rightharpoonup u_0 \quad \text{in } H^1(\mathbb{R}^N).$$

In view of the Rellich-Kondrachov theorem

$$u_m\mathbf{1}_{B(0,R)}\xrightarrow{m\to\infty}u_0\mathbf{1}_{B(0,R)} \quad \text{in } L^2(\mathbb{R}^N)$$

– see (3.24). Since function f_n satisfies condition (f1) with Kato-Rellich type function l, we obtain the following estimate

$$\begin{aligned}
&\|F_n(t_m,u_m)\mathbf{1}_{B(0,R)}-F_n(t_0,u_0)\mathbf{1}_{B(0,R)}\|_{L^2}\\
&\quad\leq\|F_n(t_m,u_m\mathbf{1}_{B(0,R)})-F_n(t_0,u_0\mathbf{1}_{B(0,R)})\|_{L^2}\\
&\quad\leq\|F_n(t_m,u_m\mathbf{1}_{B(0,R)})-F_n(t_m,u_0\mathbf{1}_{B(0,R)})\|_{L^2}\\
&\qquad+\|F_n(t_m,u_0\mathbf{1}_{B(0,R)})-F_n(t_0,u_0\mathbf{1}_{B(0,R)})\|_{L^2}\\
&\quad\leq\|Lu_m\mathbf{1}_{B(0,R)}-Lu_0\mathbf{1}_{B(0,R)}\|_{L^2}+\|F_n(t_m,u_0\mathbf{1}_{B(0,R)})-F_n(t_0,u_0\mathbf{1}_{B(0,R)})\|_{L^2}\\
&\quad\leq\|L_\infty u_m\mathbf{1}_{B(0,R)}-L_\infty u_0\mathbf{1}_{B(0,R)}\|_{L^2}+\|L_0u_m\mathbf{1}_{B(0,R)}-L_0u_0\mathbf{1}_{B(0,R)}\|_{L^2}\\
&\qquad+\|F_n(t_m,u_0\mathbf{1}_{B(0,R)})-F_n(t_0,u_0\mathbf{1}_{B(0,R)})\|_{L^2}.
\end{aligned}\tag{3.30}$$

As L_∞ is a bounded linear operator on $L^2(\mathbb{R}^N)$ (cf. Corollary 3.1.1) and F_n is continuous (cf. Proposition 3.1.2 (i)), we get that

$$\|L_\infty u_m\mathbf{1}_{B(0,R)}-L_\infty u_0\mathbf{1}_{B(0,R)}\|_{L^2}\to 0 \ \text{ as } \ m\to\infty \tag{3.31}$$

and

$$\|F_n(t_m, u_0\mathbf{1}_{B(0,R)}) - F_n(t_0, u_0\mathbf{1}_{B(0,R)})\|_{L^2} \to 0 \ \text{ as } \ m\to\infty. \tag{3.32}$$

By Proposition 2.1.5 linear operator $L_0 : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is compact. Hence,

$$L_0 u_m \mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} L_0 u_0 \mathbf{1}_{B(0,R)} \ \text{ in } \ L^2(\mathbb{R}^N) \tag{3.33}$$

because

$$u_m \mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} u_0 \mathbf{1}_{B(0,R)} \quad \text{in } H^1(\mathbb{R}^N).$$

Combining (3.31), (3.32), (3.33) with (3.30) we obtain that $(w_m)_{m\ge 1}$ admits a convergent subsequence.

We consider now the second case, that is, there are subsequences $(w_{m_k})_{k\ge 1}$ and $(F_{n_k})_{k\ge 1}$ such that (3.29) is satisfied. Obviously, we may assume that

$$w_m \in F_m(D\times U)\mathbf{1}_{B(0,R)} \quad \text{for all } m\ge 1.$$

Hence, there exist sequences $(t_m)_{m\ge 1}$ in D and $(u_m)_{m\ge 1}$ in U such that

$$w_m = F_m(t_m, u_m)\mathbf{1}_{B(0,R)} \quad \text{for all } m\ge 1.$$

As before, passing to a subsequence if necessary, $t_m \to t_0$ for some $t_0 \in \overline{D}$,

$$u_m \mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} u_0 \mathbf{1}_{B(0,R)} \quad \text{in } H^1(\mathbb{R}^N),$$

and

$$u_m \mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} u_0 \mathbf{1}_{B(0,R)} \quad \text{in } L^2(\mathbb{R}^N).$$

From (3.31) and (3.33) we deduce that

$$L u_m \mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} L u_0 \mathbf{1}_{B(0,R)} \ \text{ in } \ L^2(\mathbb{R}^N). \tag{3.34}$$

In view of the assumption, $f_m \to f_0$ in the sense of condition (f0). Therefore, Proposition 3.1.3 (ii) yields

$$F_m(t_m, u_0\mathbf{1}_{B(0,R)}) \xrightarrow{m\to\infty} F_0(t_0, u_0\mathbf{1}_{B(0,R)}) \quad \text{in } L^2(\mathbb{R}^N). \tag{3.35}$$

By assumption functions f_m, $m\ge 1$, satisfy condition (f1) with a common Kato-Rellich type function l. Hence, we get the following estimate

$$\begin{aligned} &\|F_m(t_m, u_m\mathbf{1}_{B(0,R)}) - F_0(t_0, u_0\mathbf{1}_{B(0,R)})\|_{L^2} \\ &\quad \le \|F_m(t_m, u_m\mathbf{1}_{B(0,R)}) - F_m(t_m, u_0\mathbf{1}_{B(0,R)})\|_{L^2} \\ &\qquad + \|F_m(t_m, u_0\mathbf{1}_{B(0,R)}) - F_0(t_0, u_0\mathbf{1}_{B(0,R)})\|_{L^2} \\ &\quad \le \|L u_m\mathbf{1}_{B(0,R)} - L u_0\mathbf{1}_{B(0,R)}\|_{L^2} + \|F_m(t_m, u_0\mathbf{1}_{B(0,R)}) - F_0(t_0, u_0\mathbf{1}_{B(0,R)})\|_{L^2}. \end{aligned} \tag{3.36}$$

We then apply (3.34) and (3.35) to (3.36) obtaining

$$F_m(t_m, u_m)\mathbf{1}_{B(0,R)} \xrightarrow{m\to\infty} F_0(t_0, u_0)\mathbf{1}_{B(0,R)} \quad \text{in } L^2(\mathbb{R}^N).$$

This shows that $(w_m)_{m\ge 1}$ admits a convergent subsequence, and the proof is completed. □

Proof of Proposition 3.3.1. Let $0<\varepsilon<1$ and take the covering of the set U:

$$U \subset \bigcup_{i=1}^{k} B_{L^2}(u_i, \chi_{L^2}(U)+\varepsilon).$$

As $H^1(\mathbb{R}^N)$ is a dense subspace of $L^2(\mathbb{R}^N)$, we may assume that any $u_i \in H^1(\mathbb{R}^N)$. Observe that, for any $R>0$, if l is a Kato-Rellich type function, then $l\,\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}$ is also a Kato-Rellich type function. Hence, by Corollary 3.1.1, one has, for any $R>0$ and $u\in H^1(\mathbb{R}^N)$,

$$\begin{aligned} \|Lu\|_{L^2(\mathbb{R}^N\setminus B(0,R))} &= \|(L\,\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)})u\|_{L^2} \le \|l_\infty\,\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\|\,\|u\|_{L^2} + C\|l_0\,\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\|_{L^p}\,\|u\|_{H^1} \\ &= \|l_\infty\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))}\|u\|_{L^2} + C\|l_0\|_{L^p(\mathbb{R}^N\setminus B(0,R))}\|u\|_{H^1} \end{aligned}$$

where $C = C(N, p) > 0$ is a constant. Let R be such that

$$\|l_\infty\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))} < \widehat{\varrho}(l_\infty) + \varepsilon.$$

Since C does not depend on R and $U \subset H^1(\mathbb{R}^N)$ is bounded, then, increasing R if necessary, we have

$$C\|l_0\|_{L^p(\mathbb{R}^N\setminus B(0,R))}\|u - u_i\|_{H^1} < \varepsilon \;\text{ for }\; u \in U \;\text{ and }\; i \in \{1, \dots, k\}.$$

Hence, for any $u \in U$ and $i \in \{1, \dots, k\}$,

$$\|Lu - Lu_i\|_{L^2(\mathbb{R}^N\setminus B(0,R))} \le (\widehat{\varrho}(l_\infty) + \varepsilon)\|u - u_i\|_{L^2} + \varepsilon. \tag{3.37}$$

Observe that

$$\begin{aligned}\bigcup_{n=1}^{\infty} F_n(D \times U) &\subset \bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{B(0,R)} + F_n(D \times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)} \\ &\subset \bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{B(0,R)} + \bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}.\end{aligned}$$

Hence, due to the monotonicity and algebraic semi-additivity of the measure of non-compactness (cf. Proposition A.3.1),

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\Big) \le \chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{B(0,R)}\Big) + \chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\Big). \tag{3.38}$$

By Lemma 3.3.2 the set $\bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{B(0,R)}$ is relatively compact in $L^2(\mathbb{R}^N)$, which implies

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\Big) \le \chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\Big). \tag{3.39}$$

Since $D \subset [0, +\infty)$ is relatively compact and $F_0 : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous, due to Proposition 3.1.3 (ii), for any $i \in \{1, \dots, k\}$, the sequence

$$D \ni t \mapsto F_n(t, u_i) \in L^2(\mathbb{R}^N), \quad n \ge 1,$$

converges uniformly to

$$D \ni t \mapsto F_0(t, u_i) \in L^2(\mathbb{R}^N).$$ (2)

As we have a finite number of sequences $\big(t \mapsto F_n(t, u_i)\big)_{n\ge1}$, there exists $n_\varepsilon \ge 1$ such that, for $n > n_\varepsilon$ and $i \in \{1, \dots, k\}$,

$$\|F_n(t, u_i) - F_0(t, u_i)\|_{L^2} < \varepsilon \;\text{ for } t \in D. \tag{3.40}$$

Since D is relatively compact and operators $F_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n = 0, \dots, n_\varepsilon$, are continuous, functions

$$\overline{D} \ni t \mapsto F_n(t, u_i) \in L^2(\mathbb{R}^N), \quad n = 0, 1, \dots, n_\varepsilon,\; i = 1, \dots, k,$$

are uniformly continuous. Hence, there exists a number $\delta > 0$ such that, for any t_1, $t_2 \in \overline{D}$ fulfilling $|t_1 - t_2| < \delta$, any $n \in \{0, 1, \dots, n_\varepsilon\}$ and $i \in \{1, \dots, k\}$, there holds

$$\|F_n(t_1, u_i) - F_n(t_2, u_i)\|_{L^2} < \varepsilon. \tag{3.41}$$

Now, we consider a set $\{t_j \in \overline{D} : 1 \le j \le m\}$ such that for a given $t \in \overline{D}$ there is t_j satisfying $|t - t_j| < \delta$.

(2)See footnote on p. 51.

We claim that

$$\bigcup_{n=1}^{\infty} F_n(D \times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)} \subset \bigcup_{i=1}^{k}\bigcup_{n=0}^{n_\varepsilon}\bigcup_{j=1}^{m} B_{L^2}\big(F_n(t_j,u_i),\eta\big) \tag{3.42}$$

where

$$\eta = (\widehat{\varrho}(l_\infty)+\varepsilon)(\chi_{L^2}(U)+\varepsilon)+2\varepsilon.$$

Indeed, for a given element $F_n(t,u)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}$ there exist $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,m\}$ such that $u \in B_{L^2}(u_i, \chi_{L^2}(U)+\varepsilon)$ and $|t-t_j| < \delta$. If $n \in \{1,\ldots,n_\varepsilon\}$, then, in view of (3.37) and the uniform continuity (3.41), one gets

$$\begin{aligned}
&\|F_n(t,u)-F_n(t_j,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))}\\
&\quad\le \|F_n(t,u)-F_n(t,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))} + \|F_n(t,u_i)-F_n(t_j,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))}\\
&\quad\le \|Lu-Lu_i\|_{L^2(\mathbb{R}^N\setminus B(0,R))} + \|F_n(t,u_i)-F_n(t_j,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))}\\
&\quad\le (\varepsilon+\widehat{\varrho}(l_\infty))\|u-u_i\|_{L^2}+\varepsilon+\varepsilon < (\varepsilon+\widehat{\varrho}(l_\infty))(\chi_{L^2}(U)+\varepsilon)+2\varepsilon.
\end{aligned}$$

On the other hand, if $n > n_\varepsilon$, then, again by (3.37), the uniform convergence (3.40), and the uniform continuity (3.41), we arrive at

$$\begin{aligned}
&\|F_n(t,u)-F_0(t_j,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))}\\
&\le \|F_n(t,u)-F_n(t,u_i)\|_{L^2(\mathbb{R}^N\setminus B(0,R))} + \|F_n(t,u_i)-F_0(t,u_i)\|_{L^2} + \|F_0(t,u_i)-F_0(t_j,u_i)\|_{L^2}\\
&\le \|Lu-Lu_i\|_{L^2(\mathbb{R}^N\setminus B(0,R))} + 2\varepsilon < (\widehat{\varrho}(l_\infty)+\varepsilon)(\chi_{L^2}(U)+\varepsilon)+2\varepsilon.
\end{aligned}$$

Hence, (3.42) holds, as claimed.

In view of (3.42) we get

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D\times U)\mathbf{1}_{\mathbb{R}^N\setminus B(0,R)}\Big) \le (\widehat{\varrho}(l_\infty)+\varepsilon)(\chi_{L^2}(U)+\varepsilon)+2\varepsilon.$$

This together with (3.39) yields

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n(D\times U)\Big) \le (\widehat{\varrho}(l_\infty)+\varepsilon)(\chi_{L^2}(U)+\varepsilon)+2\varepsilon.$$

As ε was arbitrary, the proof is completed. □

Remark 3.3.3. Observe that Proposition 3.3.1 (and Lemma 3.3.2) are still true if we consider Carathéodory functions $f_n : M \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \ge 0$, where (M,d) is a metric space, satisfying properly changed versions of conditions (f1), (f2) and (f0), and a relatively compact subset $D \subset M$ instead of bounded subset $D \subset [0,+\infty)$. □

In view of the remark above, from Proposition 3.3.1 we get the compactness result concerning Nemytskii operator associated with a parametrized function.

Corollary 3.3.4. *Suppose that (M,d) is a compact metric space and a Carathéodory function $h : [0,+\infty)\times\mathbb{R}^N\times\mathbb{R}\times M \to \mathbb{R}$ satisfies* (h1) *and* (h2) *(see p. 53). Then, for any bounded sets $D \subset [0,+\infty)$ and $U \subset H^1(\mathbb{R}^N)$,*

$$\chi_{L^2}(H(D\times U\times M)) \le \widehat{\varrho}(l_\infty)\chi_{L^2}(U).$$

3.4 k-set contractivity of the translation along trajectories operator

Let us recall that $\mathbb{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and in $\mathbb{X}$ we consider the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ together with the associated norm $\| \cdot \|_{\mathbb{X}}$ (see p. 29). $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is the linear operator given by (2.41), i.e.

$$\mathbb{A}(u, v) = (v, -Au - \beta v) \text{ on } D(\mathbb{A}) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$$

where $A : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined by (2.7), i.e. $Au = (-\Delta + \boldsymbol{V})u$. Here $\boldsymbol{V} : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the linear operator of multiplication by a Kato-Rellich type potential $V : \mathbb{R}^N \to \mathbb{R}$ (see (5a) and (5b)) which $L^\infty(\mathbb{R}^N)$-part has a positive asymptotic bottom (see (7)). Let us recall that the operator $\mathbb{A}$ is the generator of a C_0-semigroup $\{e^{t\mathbb{A}}\}_{t\geq 0}$, see Proposition 2.2.1 (ii).

On $\mathbb{X}$ we also consider the scalar product $\langle \cdot, \cdot \rangle_{s,V}$, given by (2.58), i.e.

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{s,V} = \langle (P + P_0)u_1, (P + P_0)u_2 \rangle_{L^2} + \langle Qu_1, Qu_2 \rangle_Q + \langle v_1 + su_1, v_2 + su_2 \rangle_{L^2}$$

together with the associated norm $\| \cdot \|_{s,V}$ given by (2.60), i.e.

$$\|(u, v)\|_{s,V}^2 = \langle (u, v), (u, v) \rangle_{s,V} = \|(P + P_0)u\|_{L^2}^2 + \|Qu\|_Q^2 + \|v + su\|_{L^2}^2$$

where the scalar product $\langle \cdot, \cdot \rangle_{s,V}$ and the norm $\| \cdot \|_{s,V}$ depend on a parameter $s > 0$ and a Kato-Rellich type potential V. Due to Theorem 2.3.3 the norms $\| \cdot \|_{\mathbb{X}}$ and $\| \cdot \|_{s,V}$ are equivalent. $\chi_{s,V}(\cdot)$ stands for the Hausdorff measure of non-compactness in $(\mathbb{X}, \| \cdot \|_{s,V})$. By virtue of Theorem 2.3.9 there exist $s > 0$ and $\rho > 0$ as in Proposition 2.3.7 such that

$$\chi_{s,V}(e^{t\mathbb{A}}(Z)) \leq e^{-\rho t} \chi_{s,V}(Z) \quad \text{for any } t > 0 \text{ and bounded } Z \subset \mathbb{X}. \tag{3.43}$$

Below we state a result on the existence and continuity of mild solutions, and the k-set contractivity of the translation along trajectories operator for the following parametrized problem

$$(\dot{u}(t), \dot{v}(t)) = \lambda \mathbb{A}(u(t), v(t)) + \big(0, H(t, u(t), \mu)\big), \quad t > 0, \ \mu \in M, \ \lambda > 0 \tag{3.44}$$

where (M, d) is a compact metric space and $H : [0, +\infty) \times H^1(\mathbb{R}^N) \times M \to L^2(\mathbb{R}^N)$ is a continuous mapping satisfying (**H1**) and (**H2**) (see p. 54). For any $\lambda > 0$, the linear operator $\lambda\mathbb{A} : D(\lambda\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is defined in a usual way:

$$(\lambda\mathbb{A})(u, v) = \lambda\mathbb{A}(u, v) \quad \text{on} \quad D(\lambda\mathbb{A}) = D(\mathbb{A}).$$

Recall that a continuous function $(u, v)(\cdot, \overline{u}, \overline{v}, \mu, \lambda) : [0, +\infty) \to \mathbb{X}$ is called the *mild solution* of (3.44) with an initial condition $(\overline{u}, \overline{v}) \in \mathbb{X}$ if it satisfies the *Duhamel formula*

$$(u(t), v(t)) = e^{t(\lambda\mathbb{A})}(\overline{u}, \overline{v}) + \int_0^t e^{(t-\tau)(\lambda\mathbb{A})}\big(0, H(\tau, u(\tau), \mu)\big)\, d\tau \quad \text{for } t \geq 0 \tag{3.45}$$

– see Section 1.2.

Theorem 3.4.1.

(i) For any $(\overline{u}, \overline{v}) \in \mathbb{X}$, $\mu \in M$ and $\lambda > 0$ there exists the unique global mild solution of (3.44) with an initial condition $(u(0), v(0)) = (\overline{u}, \overline{v})$. In particular, the translation along trajectories operator (by time $t_0 > 0$) associated with equation (3.44), $\Phi_{t_0} : \mathbb{X} \times M \times (0, +\infty) \to \mathbb{X}$, given by

$$\Phi_{t_0}(\overline{u}, \overline{v}, \mu, \lambda) = (u, v)(t_0, \overline{u}, \overline{v}, \mu, \lambda)$$

is well-defined for any $t_0 > 0$.

(ii) *If* $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$ *in* $\mathbb{X}$, $\mu_n \to \mu_0$ *in* M *and* $\lambda_n \to \lambda_0$ *for* $\lambda_0 > 0$, *then*

$$(u, v)(t, \overline{u}_n, \overline{v}_n, \mu_n, \lambda_n) \to (u, v)(t, \overline{u}_0, \overline{v}_0, \mu_0, \lambda_0)$$

uniformly on bounded subsets of $[0, +\infty)$ *with respect to* t. *In particular, the translation along trajectories operator* $\Phi_{t_0} : \mathbb{X} \times M \times (0, +\infty) \to \mathbb{X}$ *is continuous for any* $t_0 > 0$.

(iii) *Suppose additionally that there exists* $k \in [0, \sqrt{d}\,\rho)$, *where (cf.* (2.54) *on p. 32),*

$$d = \operatorname{dist}(0, \sigma(A) \cap (0, +\infty)) \tag{3.46}$$

such that, for any bounded subsets $D \subset [0, +\infty)$ *and* $U \subset H^1(\mathbb{R}^N)$,

$$\chi_{L^2}(H(D \times U \times M)) \le k\chi_{L^2}(U). \tag{3.47}$$

Then, for any $t_0 > 0$, *the translation along trajectories operator* $\Phi_{t_0} : \mathbb{X} \times M \times [a, b] \to \mathbb{X}$, *where* $0 < a \le b < +\infty$, *is a* k*-set contraction with respect to the measure of noncompactness* $\chi_{s,V}$ *(see Section A.4), more precisely, for any bounded subset* $Z \subset \mathbb{X}$,

$$\chi_{s,V}\big(\Phi_{t_0}(Z \times M \times [a, b])\big) \le e^{-(\rho - k/\sqrt{d})at_0}\chi_{s,V}(Z). \tag{3.48}$$

In particular, the thesis holds if H *is completely continuous, that is, it satisfies* (3.47) *with* $k = 0$.

Proof. (i) Since the linear operator $\mathbb{A}$ is the generator of a C_0-semigroup and $\lambda > 0$, $\lambda\mathbb{A}$ is the generator of a C_0-semigroup and $e^{t(\lambda\mathbb{A})}(u, v) = e^{(t\lambda)\mathbb{A}}(u, v)$, for all $t \ge 0$ and $(u, v) \in \mathbb{X}$. Observe that the Lipschitz condition and boundedness at zero imply that H has a sublinear growth, that is, there exists $C > 0$ such that

$$\|H(t, u, \mu)\|_{L^2} \le C(1 + \|u\|_{H^1})$$

for all $t \ge 0$, $u \in H^1(\mathbb{R}^N)$ and $\mu \in M$. Define $\mathbb{H} : [0, +\infty) \times \mathbb{X} \times M \to \mathbb{X}$ by formula $\mathbb{H}(t, u, v, \mu) = \big(0, H(t, u, \mu)\big)$. Further, let $\Pi : \mathbb{X} \to H^1(\mathbb{R}^N)$ be given by $\Pi(u, v) = u$ and $j : L^2(\mathbb{R}^N) \to \mathbb{X}$ be given by $j(v) = (0, v)$. We see that Π and j are bounded linear operators and, for any $t \ge 0$, $(u, v) \in \mathbb{X}$, $\mu \in M$,

$$\mathbb{H}(t, u, v, \mu) = j\big(H(t, \Pi(u, v), \mu)\big). \tag{3.49}$$

Hence, $\mathbb{H}$ is continuous, fulfills the Lipschitz condition and has a sublinear growth, i.e., it satisfies conditions (H1) and (H2) – see p. 15. Therefore, we are in a position to apply the well-known global existence and uniqueness result (see Section 1.5) to get that, for any $(\overline{u}, \overline{v}) \in \mathbb{X}$, $\mu \in M$ and $\lambda > 0$, there exists a unique global mild solution $(u, v)(\cdot, \overline{u}, \overline{v}, \mu, \lambda) : [0, +\infty) \to \mathbb{X}$ of (3.44) with $(u(0), v(0)) = (\overline{u}, \overline{v})$.

(ii) Let us take a sequence $\lambda_n \to \lambda_0$ for $\lambda_0 > 0$. This implies the existence of an interval $[a, b]$, where $0 < a \le b < +\infty$, such that $\lambda_n \in [a, b]$ for all $n \ge 0$. Next, we see that $M \times [a, b]$ is a compact metric space. By Remark 1.1.1 there exist numbers $K \ge 1$ and $\omega \in \mathbb{R}$ such that $\|e^{t\mathbb{A}}\|_{\mathcal{L}(\mathbb{X})} \le Ke^{\omega t}$ for $t \ge 0$. Hence, we have

$$\|e^{t(\lambda\mathbb{A})}(u, v)\|_{\mathcal{L}(\mathbb{X})} \le Ke^{\omega\lambda t} \le Ke^{|\omega|bt} \tag{3.50}$$

for all $t \ge 0$ and $\lambda \in [a, b]$. This means that the family of linear operators $\{\lambda\mathbb{A}\}_{\lambda \in [a,b]}$ fulfills condition (A1) (see p. 15). Then, using the resolvent identity(3), we get, for any $\lambda > |\omega|b$ and $(u, v) \in \mathbb{X}$,

$$\begin{aligned} R(\lambda, \lambda_n\mathbb{A})(u, v) - R(\lambda, \lambda_0\mathbb{A})(u, v) &= R(\lambda, \lambda_n\mathbb{A})\ (\lambda_n\mathbb{A} - \lambda_0\mathbb{A})\ R(\lambda, \lambda_0\mathbb{A})(u, v) \\ &= (\lambda_n - \lambda_0)\ R(\lambda, \lambda_n\mathbb{A})\ \mathbb{A}\ R(\lambda, \lambda_0\mathbb{A})(u, v). \end{aligned}$$

(3)We use the resolvent identity of the form (see [36, Prop. 1.9]): let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators on a Banach space X such that $D(A) = D(B)$; then, for any $\lambda \in \rho(A) \cap \rho(B)$,

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)\ (A - B)\ R(\lambda, B).$$

Consequently, using (3.50) and the Hille-Yosida Theorem (see Theorem 1.1.5), one has

$$\|R(\lambda, \lambda_n \mathbb{A})\|_{\mathcal{L}(\mathbb{X})} \leq K/(\lambda - |\omega| b) \ \text{ for all } \lambda > |\omega| b.$$

We arrive at

$$\|R(\lambda, \lambda_n \mathbb{A})(u,v) - R(\lambda, \lambda_0 \mathbb{A})(u,v)\| \leq |\lambda_n - \lambda_0| \, \frac{K}{\lambda - |\omega| b} \, \|\mathbb{A}\, R(\lambda, \lambda_0 \mathbb{A})(u,v)\| \to 0 \ \text{ as } n \to \infty.$$

Therefore, the family of operators $\{\lambda \mathbb{A}\}_{\lambda \in [a,b]}$ satisfies condition (A2) (see p. 15). This allows us to apply Theorem 1.5.1 (i), hence $(u,v)(t, \overline{u}_n, \overline{v}_n, \mu_n, \lambda_n) \to (u,v)(t, \overline{u}_0, \overline{v}_0, \mu_0, \lambda_0)$ uniformly on bounded subsets of $[0, +\infty)$ with respect to t.

(iii) As the $L^\infty(\mathbb{R}^N)$-part of a Kato-Rellich type potential V has a positive asymptotic bottom, from Proposition 2.3.1 (iii) we deduce that $d > 0$ and the assumption that $k \in [0, \sqrt{d}\,\rho)$ makes sense. Let $D \subset [0, +\infty)$ and $Z \subset \mathbb{X}$ be bounded. By (3.43) there exist $s > 0$ and $\rho > 0$ such that, for any $t > 0$,

$$\chi_{s,V}\big(\big\{e^{t(\lambda \mathbb{A})}(u,v) : \, (u,v) \in Z, \lambda \in [a,b]\big\}\big) \leq e^{-\rho a t} \chi_{s,V}(Z). \tag{3.51}$$

In view of (3.49) one has

$$\mathbb{H}(D \times Z \times M) = j\big(H(D \times \Pi(Z) \times M)\big).$$

Since $j : (L^2(\mathbb{R}^N), \|\cdot\|_{L^2}) \to (\mathbb{X}, \|\cdot\|_{s,V})$ is a bounded linear operator, the measure of non-compactness can be estimated as (see Proposition A.3.1 (v))

$$\begin{aligned} \chi_{s,V}\big(\mathbb{H}(D \times Z \times M)\big) &= \chi_{s,V}\big(j\big(H(D \times \Pi(Z) \times M)\big)\big) \\ &\leq \|j\|_{\mathcal{L}(L^2(\mathbb{R}^N), \mathbb{X})} \, \chi_{L^2}\big(H(D \times \Pi(Z) \times M)\big) \\ &\leq \|j\|_{\mathcal{L}(L^2(\mathbb{R}^N), \mathbb{X})} \, k \, \chi_{L^2}(\Pi(Z)). \end{aligned} \tag{3.52}$$

Observe that, for all $v \in L^2(\mathbb{R}^N)$,

$$\|j(v)\|_{s,V} = \|(0,v)\|_{s,V} = \|v\|_{L^2}.$$

Hence,

$$\|j\|_{\mathcal{L}(L^2(\mathbb{R}^N), \mathbb{X})} = 1. \tag{3.53}$$

Now, we apply the fact that $L^2(\mathbb{R}^N)$ admits the orthogonal decomposition $L^2(\mathbb{R}^N) = X_P \oplus X_0 \oplus X_Q$ with projections P, P_0 and Q (cf. Proposition 2.3.1 (i)). Thus, by algebraic semi-additivity of measure of non-compactness,

$$\chi_{L^2}(\Pi(Z)) \leq \chi_{L^2}((P + P_0)\Pi(Z)) + \chi_{L^2}(Q\Pi(Z)).$$

As $(P + P_0)\Pi(Z)$ is a bounded subset in finite-dimensional subspace $X_P \oplus X_0$, it is relatively compact, so $\chi_{L^2}((P + P_0)\Pi(Z)) = 0$ and consequently

$$\chi_{L^2}(\Pi(Z)) \leq \chi_{L^2}(Q\Pi(Z)) \leq \|Q\Pi\|_{\mathcal{L}(\mathbb{X}, L^2(\mathbb{R}^N))} \chi_{s,V}(Z). \tag{3.54}$$

Let $(u,v) \in \mathbb{X}$. Thus, $Qu \in X_Q' = H^1(\mathbb{R}^N) \cap X_Q$. In view of Lemma 2.3.4 we have $\|Qu\|_Q^2 \geq d\|Qu\|_{L^2}^2$, which yields $\|Qu\|_{L^2} \leq (1/\sqrt{d})\|Qu\|_Q$. This implies that

$$\begin{aligned} \|Q\Pi(u,v)\|_{L^2} &= \|Qu\|_{L^2} \leq (1/\sqrt{d})\|Qu\|_Q \\ &\leq (1/\sqrt{d})\Big(\|(P + P_0)u\|_{L^2}^2 + \|Qu\|_Q^2 + \|v + su\|_{L^2}^2\Big)^{1/2} = (1/\sqrt{d})\|(u,v)\|_{s,V}. \end{aligned}$$

Therefore,

$$\|Q\Pi\|_{\mathcal{L}(\mathbb{X}, L^2(\mathbb{R}^N))} \leq 1/\sqrt{d}. \tag{3.55}$$

We take together the inequality above and (3.54) to get

$$\chi_{L^2}(\Pi(Z)) \le (1/\sqrt{d})\chi_{s,V}(Z).$$

Finally, this estimate, (3.53) and (3.52) yield

$$\chi_{s,V}\big(\mathbb{H}(D \times Z \times M)\big) \le (k/\sqrt{d})\chi_{s,V}(Z).$$

As $k \in [0, \sqrt{d}\,\rho)$, this enables us to apply Theorem 1.5.1 (ii) to obtain that, for any $t_0 > 0$ and bounded $Z \subset \mathbb{X}$, the inequality (3.48) is satisfied. □

3.5 Compactness result for a sequence of frequency changing problems

Throughout this section, we assume that $\mathbb{X}$ is equipped with the scalar product $\langle\cdot,\cdot\rangle_{s,V}$ and the corresponding norm $\|\cdot\|_{s,V}$. Let us recall that, by Proposition 2.3.6 (i), the space $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$ admits the orthogonal decomposition

$$\mathbb{X} = \mathbb{X}_P \oplus \mathbb{X}_0 \oplus \mathbb{X}_Q$$

with corresponding projections $\mathbb{P}$, $\mathbb{P}_0$, $\mathbb{Q}$. In addition, subspaces $\mathbb{X}_P$, $\mathbb{X}_0$ are finite-dimensional. By virtue of Theorem 2.3.9, the following inequality holds

$$\|e^{t\mathbb{A}}\mathbb{Q}\|_{\mathcal{L}(\mathbb{X})} \le e^{-\rho t} \quad \text{for all } t > 0, \tag{3.56}$$

where $\rho > 0$ is as in (3.43).

We now state the compactness result for a sequence of frequency changing problems.

Proposition 3.5.1. *Assume that mappings $F_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \ge 1$, are continuous, satisfy conditions* (**F1**) *and* (**F2**) *(see p. 48), with common constants, and moreover assume that*

(**F3**) *F_n are T-periodic, that is, there exists $T > 0$ such that $F_n(t+T,u) = F_n(t,u)$, for all $t \ge 0$, $u \in H^1(\mathbb{R}^N)$ and $n \ge 1$.*

Suppose that there exists $k \in [0, \sqrt{d}\,\rho)$, where d is as in (3.46)*, such that*

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n([0,T] \times U)\Big) \le k\chi_{L^2}(U), \tag{3.57}$$

for any bounded $U \subset H^1(\mathbb{R}^N)$. Additionally, let $\big((\overline{u}_n, \overline{v}_n)\big)_{n\ge1}$ be a bounded sequence in $\mathbb{X}$, and $(\alpha_n)_{n\ge1}$ be a sequence in $(0,+\infty)$ such that $\alpha_n \to \alpha_0$ in $[0,+\infty)$.

If, for any $n \ge 1$, $(u_n, v_n) : [0,+\infty) \to \mathbb{X}$ is the $\alpha_n T$-periodic mild solution of

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F_n(t/\alpha_n, u(t))\big), & t > 0, \\ (u(0), v(0)) = (\overline{u}_n, \overline{v}_n), \end{cases} \tag{3.58}$$

then the set $\{(\overline{u}_n, \overline{v}_n)\}_{n\ge1}$ is relatively compact in $\mathbb{X}$.

Remark 3.5.2. Assume that mappings $F_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \ge 1$, are continuous and satisfy conditions (**F1**) and (**F2**) with common constants, and let $\big((\overline{u}_n, \overline{v}_n)\big)_{n\ge1}$ be a bounded sequence in $\mathbb{X}$, and $(\alpha_n)_{n\ge1}$ be a sequence in $(0,+\infty)$. If $(u_n, v_n) : [0,+\infty) \to \mathbb{X}$, $n \ge 1$, are the mild solutions of (3.58), then the sets

$$\{(u_n(t), v_n(t)) : n \ge 1, t \in [0,T_0]\} \quad \text{and} \quad \big\{\big(0, F_n(t/\alpha_n, u_n(t))\big) : n \ge 1, t \in [0,T_0]\big\}$$

are bounded for any bounded interval $[0, T_0] \subset [0, +\infty)$. Indeed, we readily see that the mappings $\widetilde{\mathbb{F}}_n : [0, +\infty) \times \mathbb{X} \to \mathbb{X}$, $n \geq 1$, defined by $\widetilde{\mathbb{F}}_n(t, u, v) = (0, F_n(t/\alpha_n, u))$, are continuous, and satisfy conditions (F1) and (F2) (see p. 4) with common constants. Hence, there exists a constant $C > 0$ such that

$$\|\widetilde{\mathbb{F}}_n(t, u, v)\|_{\mathbb{X}} \leq C(1 + \|(u, v)\|_{\mathbb{X}}) \tag{3.59}$$

for all $n \geq 1$, $t \geq 0$, and $(u, v) \in \mathbb{X}$. We apply then Remark 1.2.2 to get a constant $\widetilde{C} > 0$ such that

$$\|(u_n(t), v_n(t))\|_{\mathbb{X}} \leq \widetilde{C}$$

for all $n \geq 1$ and $t \in [0, T_0]$. From this inequality and (3.59) we deduce that also the set

$$\{\widetilde{\mathbb{F}}_n\big(t, u_n(t), v_n(t)\big) : n \geq 1, t \in [0, T_0]\}$$

is bounded. □

Before the proof we need the following lemma.

Lemma 3.5.3. *Suppose that all the assumptions of Proposition 3.5.1 are satisfied and* $(t_n)_{n\geq 1}$ *is a sequence in* $(0, +\infty)$ *such that* $t_n \to t_0$ *with* $t_0 > 0$. *Then we have inequality*

$$\begin{aligned} \chi_{s,V}\big(\{(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) &\leq e^{-\rho t_0} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ &\quad + \frac{k}{\sqrt{d}} \int_0^{t_0} e^{-\rho(t_0-\tau)} \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big)\, d\tau. \end{aligned}$$

Proof. Let T_0 be a number such that $\{t_n\}_{n\geq 1} \subset [0, T_0]$. We have, by the semi-additivity and regularity of the Hausdorff measure of non-compactness $\chi_{s,V}$ (cf. Proposition A.3.1 (iii), (i)),

$$\begin{aligned} \chi_{s,V}\big(\{(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) &\leq \chi_{s,V}\big(\{(\mathbb{P} + \mathbb{P}_0)(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) \\ &\quad + \chi_{s,V}\big(\{\mathbb{Q}(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) \\ &= \chi_{s,V}\big(\{\mathbb{Q}(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) \end{aligned} \tag{3.60}$$

since $\mathbb{X} = \mathbb{X}_P \oplus \mathbb{X}_0 \oplus \mathbb{X}_Q$ and $\mathbb{X}_P$, $\mathbb{X}_0$ are finite-dimensional. By Proposition 2.3.6 (iii), $\mathbb{Q}$ and $e^{t\mathbb{A}}$ commute for $t \geq 0$. Therefore, the Duhamel formula (see (3.45)) entails

$$\begin{aligned} \chi_{s,V}\big(\{\mathbb{Q}(u_n(t_n), v_n(t_n))\}_{n\geq 1}\big) &\leq \chi_{s,V}\big(\{e^{t_n\mathbb{A}}\, \mathbb{Q}(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ &\quad + \chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\, d\tau\right\}_{n\geq 1}\right) \end{aligned} \tag{3.61}$$

where functions $J_n : [0, t_n] \to \mathbb{X}$, $n \geq 1$, are given by

$$J_n(\tau) = e^{(t_n-\tau)\mathbb{A}}\, \mathbb{Q}\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)$$

and mappings $\widetilde{F}_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \geq 1$, are defined as $\widetilde{F}_n(t, u) = F_n(t/\alpha_n, u)$. Since the sequence $(t_n)_{n\geq 1}$ is bounded and $\{e^{t\mathbb{A}}\}_{t\geq 0}$ is a C_0-semigroup (cf. Remark 1.1.1), the set $\{e^{t_n\mathbb{A}}(u, v)\}_{n\geq 1}$ is relatively compact for any $(u, v) \in \mathbb{X}$. Then the sequence of bounded linear operators estimate for $\chi_{s,V}$ from Lemma A.3.2, yields

$$\begin{aligned} \chi_{s,V}\big(\{e^{t_n\mathbb{A}}\, \mathbb{Q}\,(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) &\leq \Big(\limsup_{n\to\infty} \|e^{t_n\mathbb{A}}\, \mathbb{Q}\|_{\mathcal{L}(\mathbb{X})}\Big)\, \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ &\leq e^{-\rho t_0}\, \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \end{aligned} \tag{3.62}$$

because, in view of (3.56), $\|e^{t_n\mathbb{A}}\mathbb{Q}\|_{\mathcal{L}(\mathbb{X})} \leq e^{-\rho t_n}$ for $n \geq 1$.

We set some $\varepsilon > 0$ such that $t_0 - \varepsilon \geq 0$. Then, there exists $n_0 \geq 1$ such that $|t_n - t_0| < \varepsilon$ for all $n \geq n_0$. In particular, $t_n > t_0 - \varepsilon$ for $n \geq n_0$. Taking into account the fact that finite sets are always compact, we arrive at

$$\begin{aligned}\chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq 1}\right) &= \chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right)\\ &\leq \chi_{s,V}\left(\left\{\int_0^{t_0-\varepsilon} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right)\\ &\quad + \chi_{s,V}\left(\left\{\int_{t_0-\varepsilon}^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right).\end{aligned} \tag{3.63}$$

Using again the inequality (3.56), we get, for any $n \geq 1$ and all $\tau \in [0, t_n]$,

$$\|J_n(\tau)\|_{s,V} \leq \left\|e^{(t_n-\tau)\mathbb{A}}\mathbb{Q}\right\|_{\mathcal{L}(\mathbb{X})} \left\|\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\|_{s,V} \leq \left\|\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\|_{s,V}.$$

By Remark 3.5.2, there exists a constant $C > 0$ such that $\left\|\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\|_{s,V} \leq C$, for all $n \geq 1$ and $\tau \in [0, T_0]$. Hence, $\|J_n(\tau)\|_{s,V} \leq C$, for all $n \geq 1$ and $\tau \in [0, t_n]$. This allows us to use the integral estimate for $\chi_{s,V}$ (see Theorem A.3.3)

$$\chi_{s,V}\left(\left\{\int_0^{t_0-\varepsilon} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right) \leq \int_0^{t_0-\varepsilon} \chi_{s,V}\left(\left\{J_n(\tau)\right\}_{n\geq n_0}\right) d\tau.$$

We apply again the sequence of bounded linear operators estimate for the measure of non-compactness obtaining, for any $\tau \in [0, t_0 - \varepsilon]$,

$$\chi_{s,V}\left(\left\{J_n(\tau)\right\}_{n\geq n_0}\right) \leq e^{-\rho(t_0-\tau)}\chi_{s,V}\left(\left\{\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\}_{n\geq n_0}\right).$$

From this we deduce that

$$\chi_{s,V}\left(\left\{\int_0^{t_0-\varepsilon} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right) \leq \int_0^{t_0-\varepsilon} e^{-\rho(t_0-\tau)}\chi_{s,V}\left(\left\{\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\}_{n\geq n_0}\right) d\tau. \tag{3.64}$$

Note that, as $t_n - (t_0 - \varepsilon) = (t_n - t_0) + \varepsilon < 2\varepsilon$, for all $n \geq n_0$,

$$\left\|\int_{t_0-\varepsilon}^{t_n} J_n(\tau)\,d\tau\right\|_{s,V} \leq 2C\varepsilon.$$

Hence,

$$\chi_{s,V}\left(\left\{\int_{t_0-\varepsilon}^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq n_0}\right) \leq 2C\varepsilon,$$

which, together with (3.63) and (3.64), yields

$$\begin{aligned}\chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq 1}\right) &\leq \int_0^{t_0-\varepsilon} e^{-\rho(t_0-\tau)}\chi_{s,V}\left(\left\{\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\}_{n\geq n_0}\right) d\tau + 2C\varepsilon\\ &\leq \int_0^{t_0-\varepsilon} e^{-\rho(t_0-\tau)}\chi_{s,V}\left(\left\{\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\}_{n\geq 1}\right) d\tau + 2C\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we arrive at

$$\chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\,d\tau\right\}_{n\geq 1}\right) \leq \int_0^{t_0} e^{-\rho(t_0-\tau)}\chi_{s,V}\left(\left\{\left(0, \widetilde{F}_n(\tau, u_n(\tau))\right)\right\}_{n\geq 1}\right) d\tau. \tag{3.65}$$

We claim that, for any $\tau \in [0, t_0]$,

$$\chi_{s,V}\Big(\big\{\big(0, \widetilde{F}_n(\tau, u_n(\tau))\big)\big\}_{n\geq 1}\Big) \leq \frac{k}{\sqrt{d}}\, \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big). \tag{3.66}$$

Note that

$$\big\{\big(0, \widetilde{F}_n(\tau, u_n(\tau))\big)\big\}_{n\geq 1} = j\big(\{\widetilde{F}_n(\tau, u_n(\tau))\}_{n\geq 1}\big) \tag{3.67}$$

where $j : (L^2(\mathbb{R}^N), \|\cdot\|_{L^2}) \to (\mathbb{X}, \|\cdot\|_{s,V})$ is given as $j(v) = (0, v)$. The map j is a bounded linear operator satisfying $\|j\|_{\mathcal{L}(L^2(\mathbb{R}^N),\mathbb{X})} = 1$ (see (3.53)). Let

$$U = \{u_n(\tau)\}_{n\geq 1}.$$

By Remark 3.5.2, the set U is bounded in $H^1(\mathbb{R}^N)$. Since operators $F_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are T-periodic, i.e., they fulfill condition (**F3**), one has

$$\big\{\widetilde{F}_n(\tau, u_n(\tau))\big\}_{n\geq 1} \subset \bigcup_{n=1}^{\infty} F_n([0, T] \times U). \tag{3.68}$$

By (3.57), (3.67) and (3.68) we get

$$\begin{aligned}\chi_{s,V}\Big(\big\{\big(0, \widetilde{F}_n(\tau, u_n(\tau))\big)\big\}_{n\geq 1}\Big) &\leq \|j\|_{\mathcal{L}(L^2(\mathbb{R}^N),\mathbb{X})}\, \chi_{L^2}\big(\{\widetilde{F}_n(\tau, u_n(\tau))\}_{n\geq 1}\big)\\ &\leq k\chi_{L^2}(U).\end{aligned} \tag{3.69}$$

As subspaces X_P and X_0 are finite-dimensional, we see that

$$\chi_{L^2}(U) = \chi_{L^2}((P + P_0 + Q)U) \leq \chi_{L^2}(QU).$$

Since

$$U = \Pi\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big)$$

where $\Pi : \mathbb{X} \to H^1(\mathbb{R}^N)$, $\Pi(u, v) = u$ for $(u, v) \in \mathbb{X}$, and the linear operator $Q\Pi : (\mathbb{X}, \|\cdot\|_{s,V}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{L^2})$ is bounded and fulfills $\|Q\Pi\|_{\mathcal{L}(\mathbb{X},L^2(\mathbb{R}^N))} \leq 1/\sqrt{d}$ (cf. (3.55)), we arrive at

$$\chi_{L^2}(U) \leq \|Q\Pi\|_{\mathcal{L}(\mathbb{X},L^2(\mathbb{R}^N))}\, \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big) \leq \frac{1}{\sqrt{d}}\, \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big).$$

From this and (3.69) we get (3.66), as desired.

Finally, from (3.65) and (3.66) we deduce that

$$\chi_{s,V}\left(\left\{\int_0^{t_n} J_n(\tau)\, d\tau\right\}_{n\geq 1}\right) \leq \frac{k}{\sqrt{d}} \int_0^{t_0} e^{-\rho(t_0-\tau)} \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big)\, d\tau.$$

Therefore, taking together (3.60), (3.61), (3.62) and the estimate above, we get the desired inequality. □

Proof of Proposition 3.5.1. Let $t > 0$. Applying Lemma 3.5.3 for a constant sequence $t_n = t$ we get

$$\begin{aligned}\chi_{s,V}\big(\big\{(u_n(t), v_n(t))\big\}_{n\geq 1}\big) &\leq e^{-\rho t} \chi_{s,V}\big(\big\{(\overline{u}_n, \overline{v}_n)\big\}_{n\geq 1}\big)\\ &\quad + \frac{k}{\sqrt{d}} \int_0^t e^{-\rho(t-\tau)} \chi_{s,V}\big(\big\{(u_n(\tau), v_n(\tau))\big\}_{n\geq 1}\big)\, d\tau.\end{aligned}$$

Let function $g : [0, +\infty) \to [0, +\infty)$ be given by

$$g(\tau) = \chi_{s,V}\big(\{(u_n(\tau), v_n(\tau))\}_{n\geq 1}\big).$$

Based on the inequality above and the Gronwall inequality, we obtain that

$$g(t) \leq e^{-(\rho - k/\sqrt{d})t} g(0)$$

that is,

$$\chi_{s,V}\big(\{(u_n(t), v_n(t))\}_{n\geq 1}\big) \leq e^{-(\rho - k/\sqrt{d})t} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big). \tag{3.70}$$

We define the sequence $(t_n)_{n\geq 1}$ as follows

$$t_n = \begin{cases} \alpha_n T, & \text{if } \alpha_0 > 0, \\ ([1/\alpha_n] + 1)\alpha_n T, & \text{if } \alpha_0 = 0. \end{cases}$$

Observe that, if $\alpha_0 > 0$, then $t_n \to \alpha_0 T$ and, if $\alpha_0 = 0$, then $T < t_n \leq T + \alpha_n T$, for all $n \geq 1$, hence $t_n \to T$. Hence, $t_n \to t_0$ where $t_0 > 0$. By assumption, mild solutions $(u_n, v_n) : [0, +\infty) \to \mathbb{X}$, $n \geq 1$, are $\alpha_n T$-periodic, hence $(\overline{u}_n, \overline{v}_n) = (u_n(t_n), v_n(t_n))$ for all $n \geq 1$. Thus, Lemma 3.5.3 yields

$$\begin{aligned} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \leq{} & e^{-\rho t_0} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ & + \frac{k}{\sqrt{d}} \int_0^{t_0} e^{-\rho(t_0 - \tau)} \chi_{s,V}\left(\big\{(u_n(\tau), v_n(\tau))\big\}_{n\geq 1}\right) d\tau. \end{aligned}$$

We use (3.70) for the estimate above to get that

$$\begin{aligned} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) &\leq e^{-\rho t_0} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ &\quad + \frac{k}{\sqrt{d}} \int_0^{t_0} e^{-\rho(t_0 - \tau)} e^{-(\rho - k/\sqrt{d})\tau} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big)\, d\tau \\ &= \Big(e^{-\rho t_0} + \frac{k}{\sqrt{d}} \int_0^{t_0} e^{-\rho(t_0 - \tau)} e^{-(\rho - k/\sqrt{d})\tau}\, d\tau\Big) \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) \\ &= e^{-(\rho - k/\sqrt{d})t_0} \chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big). \end{aligned}$$

As $k \in [0, \sqrt{d}\,\rho)$ and $t_0 > 0$, we obtain that $\chi_{s,V}\big(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}\big) = 0$, i.e., $\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}$ is relatively compact in $\mathbb{X}$. The proof is finished. □

Chapter 4

Periodic solutions for equations at resonance

In this chapter, we study periodic solutions of the nonlinear damped wave equation (1), namely,

$$u_{tt} + \beta u_t = \Delta u - V(x)u + f(t,x,u), \quad t > 0, \ x \in \mathbb{R}^N, \tag{4.1}$$

in the resonant case, that is, when the resonance condition (4) holds:

$$X_0 = \operatorname{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\} \quad \text{and} \quad f \text{ is bounded by a square integrable function,} \tag{4.2}$$

i.e., f satisfies condition (f2)$'$ – see p. ix. We also assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type potential (cf. (5a) and (5b)) such that the asymptotic bottom of the $L^\infty(\mathbb{R}^N)$-part of V is positive (cf. (7)). Moreover, we assume that $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a T-periodic Carathéodory function satisfying condition (f1) (see condition (P) on p. ix, condition (C) on p. viii, and condition (f1) on p. ix, respectively).

We seek T-periodic mild solutions of (4.1), i.e., mild solutions of the evolution equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F(t, u(t))\big), \quad t > 0 \tag{4.3}$$

where $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is the damped wave operator (see Section 2.2), and $F : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the function f (see Section 3.1).

In Section 4.1, we compute the topological fixed-point index of the translation operator associated with autonomous evolution equations. A version of the resonant averaging principle, combined with the continuation principle, is proved in Section 4.2.

In Section 4.3, we focus on geometric conditions imposed on the nonlinear term – namely, the Landesman-Lazer and strong resonance conditions. Based on these, we derive new geometric conditions, along with index formulae on the finite-dimensional space X_0.

In Section 4.4, we combine these results with the continuation principle to prove the main theorem stating that the equation (1) admits T-periodic mild solution. Finally, in Section 4.5, we apply this theorem to the equation (1) with the Coulomb potential and specific nonlinearities.

4.1 Index formula for autonomous equations

Let us recall that, by Proposition 2.3.1 (i) applied for $A = -\Delta + \boldsymbol{V}$, the space $L^2(\mathbb{R}^N)$ admits the orthogonal decompositions:

$$L^2(\mathbb{R}^N) = X_P \oplus X_0 \oplus X_Q$$

with projections denoted by P, P_0, and Q on corresponding subspaces. Then

$$L^2(\mathbb{R}^N) = X_0 \oplus \widetilde{X}, \quad \text{where} \quad \widetilde{X} = X_P \oplus X_Q,$$

with corresponding projections P_0 and $\widetilde{P}$. In addition, by Proposition 2.3.1 (ii), the space $H^1(\mathbb{R}^N)$ decomposes as the direct topological sums:

$$H^1(\mathbb{R}^N) = X_P \oplus X_0 \oplus X'_Q, \quad \text{where } X'_Q = H^1(\mathbb{R}^N) \cap X_Q,$$

and

$$H^1(\mathbb{R}^N) = X_0 \oplus \widetilde{X}', \quad \text{where } \widetilde{X}' = H^1(\mathbb{R}^N) \cap \widetilde{X}.$$

In this section we derive a topological index formula for the translation along trajectories operator associated with the autonomous equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon\big(0, F_0(P_0 u(t))\big), \quad t > 0 \tag{4.4}$$

where $\varepsilon \in (0,1]$ and $F_0 : X_0 \to X_0$ is a nonlinear perturbation.

Theorem 4.1.1. *Suppose that $F_0 : X_0 \to X_0$ satisfies the Lipschitz condition (see* (F1) *on p. 4), r, $R > 0$ and $U \subset X_0$ is an open bounded set such that $F_0(u) \neq 0$ for $u \in \partial U$. Let, for any $\varepsilon \in (0,1]$, $\overline{\Phi}_T^{(\varepsilon)} : \mathbb{X} \to \mathbb{X}$ be the translation operator associated with* (4.4) *and $\mathbb{U} = (U \oplus \widetilde{B}'_r) \times B_R$, where $\widetilde{B}'_r = B_{\widetilde{X}'}(0,r)$ and $B_R = B_{L^2}(0,R)$.*

Then, there exists $\varepsilon_0 \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\mathrm{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}) = (-1)^{m_-(-\Delta + \boldsymbol{V}) + \dim X_0} \deg_B(F_0, U)$$

where $m_-(-\Delta + \boldsymbol{V})$ is the sum of the multiplicities of the negative eigenvalues of $-\Delta + \boldsymbol{V}$[1] *and $\deg_B(F_0, U)$ is the Brouwer topological degree of the map F_0 on the set U.*

The following lemma will be used in the proof of above theorem.

Lemma 4.1.2. *Suppose that $F : E \to E$, where E is a finite-dimensional Banach space, satisfies the Lipschitz condition and $\beta > 0$. Let $\Theta_T^{(\varepsilon)} : E \times E \to E \times E$ be the translation along trajectories operator (by time $T > 0$) associated with the system*

$$\begin{cases} \dot{u}(t) = v(t), & t > 0, \\ \dot{v}(t) = -\beta v(t) + \varepsilon F(u(t)), & t > 0, \end{cases}$$

where $\varepsilon \in (0,1]$. If $U \subset E$ is an open bounded set such that $F(u) \neq 0$, for $u \in \partial U$, then, for any $R > 0$, there exists $\varepsilon_0 \in (0,1]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\Theta_T^{(\varepsilon)}(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \text{for all } (\overline{u}, \overline{v}) \in \partial(U \times B_E(0,R))$$

and

$$\mathrm{Ind}_B(\Theta_T^{(\varepsilon)}, U \times B_E(0,R)) = (-1)^{\dim E} \deg_B(F, U)$$

where $\mathrm{Ind}_B(\cdot,\cdot)$ denotes the Brouwer topological index in the space $E \times E$.

Proof. Let us consider the system

$$\begin{cases} \dot{u}(t) = (\mu + (1-\mu)\varepsilon) v(t), & t > 0, \\ \dot{v}(t) = -\mu\beta v(t) + \varepsilon F(u(t)), & t > 0, \end{cases} \tag{4.5}$$

where $\varepsilon \in (0,1]$ and $\mu \in [0,1]$ are parameters. We define $\Gamma_T^{(\varepsilon)} : E \times E \times [0,1] \to E \times E$ as the translation along trajectories operator associated with the system (4.5). We shall show that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\Gamma_T^{(\varepsilon)}(\overline{u}, \overline{v}, \mu) \neq (\overline{u}, \overline{v}), \quad \text{for all} \quad (\overline{u}, \overline{v}) \in \partial(U \times B_E(0,R)), \ \mu \in [0,1]. \tag{4.6}$$

[1]Proposition 2.1.12 (iii) shows that $\sigma(-\Delta + \boldsymbol{V}) \cap (-\infty, 0)$ consists of the finite number of eigenvalues with finite multiplicities.

If we suppose to the contrary that (4.6) does not hold, then we infer that exist $\varepsilon_n \in (0,1]$, $\mu_n \in [0,1]$ and $(\overline{u}_n, \overline{v}_n) \in \partial(U \times B_E(0,R))$, $n \geq 1$, such that $\varepsilon_n \to 0^+$ and $\Gamma_T^{(\varepsilon_n)}(\overline{u}_n, \overline{v}_n, \mu_n) = (\overline{u}_n, \overline{v}_n)$, for $n \geq 1$. Hence, for any $n \geq 1$, there exists a T-periodic solution $(u_n, v_n) : [0, +\infty) \to E \times E$ of (4.5) with $\varepsilon = \varepsilon_n$, $\mu = \mu_n$ and the initial condition $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$. Using compactness argument, we may assume that $\mu_n \to \mu_0$ and $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$, for some $\mu_0 \in [0,1]$ and

$$(\overline{u}_0, \overline{v}_0) \in \partial(U \times B_E(0,R)) = (\partial U \times \overline{B_E(0,R)}) \cup (\overline{U} \times \partial B_E(0,R)). \tag{4.7}$$

By the parameter continuity of solutions for ODEs, $((u_n, v_n))_{n\geq 1}$ converges, uniformly on bounded subsets of $[0, +\infty)$, to the T-periodic solution $(u_0, v_0) : [0, +\infty) \to E \times E$ of the system

$$\begin{cases} \dot{u}(t) = \mu_0 v(t), & t > 0, \\ \dot{v}(t) = -\mu_0 \beta v(t), & t > 0. \end{cases} \tag{4.8}$$

It follows immediately that, if $\mu_0 = 0$, then $u_0(t) = \overline{u}_0$ and $v_0(t) = \overline{v}_0$, for all $t \in [0, +\infty)$, and, if $\mu_0 \in (0,1]$, then, by the second equation of (4.8) and periodicity of v_0, $v_0(t) = \overline{v}_0 = 0$ and $u_0(t) = \overline{u}_0$, for all $t \in [0, +\infty)$.

Observe also that

$$\beta \dot{u}_n(t) + \dot{v}_n(t) = \beta(1-\mu_n)\varepsilon_n v_n(t) + \varepsilon_n F(u_n(t)), \quad \text{for} \quad n \geq 1 \text{ and } t \in [0, +\infty),$$

which, after integrating from 0 to T and dividing by ε_n, yields

$$0 = \beta(1-\mu_n) \int_0^T v_n(t)\, dt + \int_0^T F(u_n(t))\, dt \;\text{ for } n \geq 1.$$

Passing to the limits, we get

$$0 = \beta(1-\mu_0)\overline{v}_0 + F(\overline{u}_0). \tag{4.9}$$

If $\mu_0 \in (0,1]$, then, since $\overline{v}_0 = 0$ and (4.7) holds, we get $\overline{u}_0 \in \partial U$ and $F(\overline{u}_0) = 0$, a contradiction. If $\mu_0 = 0$, it follows from the second equation in (4.5) and the T-periodicity of v_n that

$$\overline{v}_n(1 - e^{-\mu_n \beta T}) = \varepsilon_n \int_0^T e^{-\mu_n \beta (T-t)} F(u_n(t))\, dt. \tag{4.10}$$

If there exists n_0 such that, for all $n \geq n_0$, $\mu_n = 0$, then (4.10) implies that, for any $n \geq n_0$, $\int_0^T F(u_n(t))\, dt = 0$ and passing to the limit we obtain $F(\overline{u}_0) = 0$. Applying (4.9) we get $\overline{v}_0 = 0$ and thus $\overline{u}_0 \in \partial U$, a contradiction. Now, we may assume that, passing to a subsequence if necessary, $\mu_n > 0$, for $n \geq 1$. Dividing (4.10) by $\mu_n T$, we get

$$\overline{v}_n \cdot \frac{1 - e^{-\mu_n \beta T}}{\mu_n T} = \frac{\varepsilon_n}{\mu_n} \cdot \frac{1}{T} \int_0^T e^{-\mu_n \beta(T-t)} F(u_n(t))\, dt \;\text{ for } n \geq 1.$$

The left-hand side of the above equation converges to $\beta \overline{v}_0$. Hence, if $((\varepsilon_n/\mu_n))_{n \geq 1}$ is unbounded, then $F(\overline{u}_0) = 0$ and, together with (4.9), we get $\overline{v}_0 = 0$, i.e., a contradiction as before. If the sequence $((\varepsilon_n/\mu_n))_{n\geq 1}$ is bounded, by passing to a subsequence if necessary, we may assume that $\varepsilon_n/\mu_n \to \delta \geq 0$, then we get $\beta \overline{v}_0 = \delta F(\overline{u}_0)$, which, again in view of (4.9), yields $\overline{v}_0 = 0 = F(\overline{u}_0)$, i.e., a contradiction completing the proof of (4.6).

Now, by use of (4.6), we apply the homotopy invariance of the topological index (see Theorem A.4.2), to get

$$\begin{aligned} \operatorname{Ind}_B(\Theta_T^{(\varepsilon)}, U \times B_E(0,R)) &= \operatorname{Ind}_B(\Gamma_T^{(\varepsilon)}(\cdot, 1), U \times B_E(0,R)) \\ &= \operatorname{Ind}_B(\Gamma_T^{(\varepsilon)}(\cdot, 0), U \times B_E(0,R)). \end{aligned} \tag{4.11}$$

Observe that, for any $\varepsilon \in (0,1]$, $\Gamma_T^{(\varepsilon)}(\cdot,0) = \Phi_{\varepsilon T}$, where $\Phi_{\varepsilon T}$ is the translation by time εT for the system

$$\begin{cases} \dot{u}(t) = v(t), & t > 0, \\ \dot{v}(t) = F(u(t)), & t > 0. \end{cases}$$

Hence, by the Krasnosel'skii Theorem (see [42, Lemma 13.1]), decreasing ε_0 if necessary, we get, for all $\varepsilon \in (0,\varepsilon_0]$,

$$\mathrm{Ind}_B(\Gamma_T^{(\varepsilon)}(\cdot,0), U \times B_E(0,R)) = \mathrm{Ind}_B(\Phi_{\varepsilon T}, U \times B_E(0,R)) = \deg_B(-\mathbf{F}, U \times B_E(0,R)) \quad (4.12)$$

where the vector field $\mathbf{F} : E \times E \to E \times E$ is given by $\mathbf{F}(u,v) = (v, F(u))$. Let us observe that

$$\deg_B(-\mathbf{F}, U \times B_E(0,R)) = (-1)^{2\dim E} \deg_B(\mathbf{F}, U \times B_E(0,R)). \quad (4.13)$$

Consider the homotopy $H : E \times E \times [0,1] \to E \times E$ given by the formula

$$H(u,v,\mu) = ((1-\mu)v - \mu F(u), (1-\mu)F(u) + \mu v).$$

We claim that $H(\overline{u},\overline{v},\mu) \neq 0$, for any $(\overline{u},\overline{v}) \in \partial(U \times B_E(0,R))$ and $\mu \in [0,1]$. Suppose, by contradiction, that $H(\overline{u},\overline{v},\mu) = 0$, for some $(\overline{u},\overline{v}) \in \partial(U \times B_E(0,R))$ and $\mu \in [0,1]$. This means that

$$\begin{bmatrix} 1-\mu & -\mu \\ \mu & 1-\mu \end{bmatrix} \begin{bmatrix} \overline{v} \\ F(\overline{u}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\det \begin{bmatrix} 1-\mu & -\mu \\ \mu & 1-\mu \end{bmatrix} = (1-\mu)^2 + \mu^2 > 0$, then $\overline{v} = 0 = F(\overline{u})$. Hence, $\overline{u} \in \partial U$, a contradiction. Then the homotopy invariance of the Brouwer degree (see Theorem A.4.1) yields

$$\deg_B(\mathbf{F}, U \times B_E(0,R)) = \deg_B(H(\cdot,0), U \times B_E(0,R)) = \deg_B(H(\cdot,1), U \times B_E(0,R))$$

where $H(u,v,1) = (-F(u), v) = (-F \times I_E)(u,v)$. Hence, by the multiplicativity of the Brouwer degree (cf. Theorem A.4.1),

$$\deg_B(H(\cdot,1), U \times B_E(0,R)) = \deg_B(-F,U)\deg_B(I_E, B_E(0,R)) = (-1)^{\dim E}\deg_B(F,U),$$

which together with (4.11), (4.12), (4.13) and the two equations above ends the proof. □

Proof of Theorem 4.1.1. Observe that the mapping $F_0 \circ P_0 : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ satisfies (**F1**). Thus, in view of Theorem 3.4.1 (points (i) and (ii)), for any $t > 0$ and $\varepsilon \in (0,1]$, the translation along trajectories operator $\overline{\Phi}_t^{(\varepsilon)}$ is well-defined and continuous. As $F_0 \circ P_0$ maps bounded subsets of $H^1(\mathbb{R}^N)$ into bounded subsets of $L^2(\mathbb{R}^N)$, and the space X_0 is finite-dimensional, this mapping is completely continuous. Then Theorem 3.4.1 (iii) shows, for any $t > 0$ and $\varepsilon \in (0,1]$, that $\overline{\Phi}_t^{(\varepsilon)}$ is a k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$ on the space $(\mathbb{X}, \|\cdot\|_{s,V})$. The norm $\|\cdot\|_{s,V}$ is induced by the scalar product $\langle\cdot,\cdot\rangle_{s,V}$ given by (2.58), and it follows from Theorem 2.3.3 that $\|\cdot\|_{s,V}$ and $\|\cdot\|_{\mathbb{X}}$ are equivalent norms.

Recall that

$$\mathbb{X}_P = X_P \times X_P, \quad \mathbb{X}_0 = X_0 \times X_0, \quad \text{and} \quad \mathbb{X}_Q = X_Q' \times X_Q,$$

and, by Proposition 2.3.6 (i), the space $(\mathbb{X}, \langle\cdot,\cdot\rangle_{s,V})$ admits the orthogonal decomposition

$$\mathbb{X} = \mathbb{X}_P \oplus \mathbb{X}_0 \oplus \mathbb{X}_Q$$

with corresponding projections $\mathbb{P}$, $\mathbb{P}_0$, $\mathbb{Q}$. Furthermore, $\widetilde{\mathbb{X}} = \mathbb{X}_P \oplus \mathbb{X}_Q$ with the associated projection $\widetilde{\mathbb{P}}$. Observe that $\widetilde{\mathbb{X}} = \widetilde{X}' \times \widetilde{X}$.

For any $\varepsilon \in (0,1]$, we define $\Theta_t^{(\varepsilon)} : \mathbb{X}_0 \to \mathbb{X}_0$, as the translation operator (by time $t > 0$) associated with the system

$$(\dot{u}(t), \dot{v}(t)) = \big(v(t), -\beta v(t) + \varepsilon F_0(u(t))\big), \quad t > 0, \tag{4.14}$$

and $\Gamma_t : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ as the translation operator associated with

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)), \quad t > 0. \tag{4.15}$$

Since $\mathbb{X}_0$ is finite-dimensional and $F_0 : X_0 \to X_0$ satisfies the Lipschitz condition, the operator $\Theta_t^{(\varepsilon)}$ is well-defined and continuous, for any $\varepsilon \in (0,1]$ and $t > 0$. By Remark 2.4.2, $\widetilde{\mathbb{X}}$ is invariant under $e^{t\mathbb{A}}$ for $t \geq 0$, hence, Γ_t is well-defined for $t > 0$. Based on Theorem 2.4.4 (i), operator Γ_t is a continuous k-set contraction with respect to $\chi_{s,V,\widetilde{\mathbb{X}}}$, for any $t > 0$. We claim that

$$\overline{\Phi}_t^{(\varepsilon)}(\overline{u}, \overline{v}) = \Theta_t^{(\varepsilon)}(\mathbb{P}_0(\overline{u}, \overline{v})) + \Gamma_t(\widetilde{\mathbb{P}}(\overline{u}, \overline{v})) \ \text{ for all } \varepsilon \in (0,1], \ (\overline{u}, \overline{v}) \in \mathbb{X}, \ t > 0. \tag{4.16}$$

Indeed, let $(u, v) : [0, +\infty) \to \mathbb{X}$ be the mild solution of (4.4) with an initial condition $(u(0), v(0)) = (\overline{u}, \overline{v})$ and define $(u_0, v_0) : [0, +\infty) \to \mathbb{X}_0$, $(u_0(t), v_0(t)) = \mathbb{P}_0(u(t), v(t))$, and $(\widetilde{u}, \widetilde{v}) : [0, +\infty) \to \widetilde{\mathbb{X}}$, $(\widetilde{u}(t), \widetilde{v}(t)) = \widetilde{\mathbb{P}}(u(t), v(t))$. Then, for all $t > 0$,

$$(u(t), v(t)) = (u_0(t), v_0(t)) + (\widetilde{u}(t), \widetilde{v}(t)). \tag{4.17}$$

Therefore, it suffices to prove that (u_0, v_0) solves (4.14) with the initial condition $(u(0), v(0)) = \mathbb{P}_0(\overline{u}, \overline{v})$ and $(\widetilde{u}, \widetilde{v})$ is the mild solution of (4.15) with the initial condition $(u(0), v(0)) = \widetilde{\mathbb{P}}(\overline{u}, \overline{v})$. Using the Duhamel formula for the solution (u, v) of (4.4) and the fact that, for $t \geq 0$, $\mathbb{P}_0$ and $e^{t\mathbb{A}}$ commute (see Proposition 2.3.6 (iii)), we obtain

$$\begin{aligned} (u_0(t), v_0(t)) &= e^{t\mathbb{A}}\mathbb{P}_0(\overline{u}, \overline{v}) + \int_0^t e^{(t-\tau)\mathbb{A}}\, \varepsilon\, \mathbb{P}_0\big(0, F_0(P_0 u(\tau))\big)\, d\tau \\ &= e^{t\mathbb{A}}\mathbb{P}_0(\overline{u}, \overline{v}) + \int_0^t e^{(t-\tau)\mathbb{A}}\, \varepsilon\big(0, F_0(u_0(\tau))\big)\, d\tau. \end{aligned}$$

This means that (u_0, v_0) is the mild solution of

$$(\dot{u}_0(t), \dot{v}_0(t)) = \mathbb{A}(u_0(t), v_0(t)) + \varepsilon\big(0, F_0(u_0(t))\big), \quad t > 0.$$

Moreover, for $t \geq 0$, we get $\mathbb{A}(u_0(t), v_0(t)) = (v_0(t), -\beta v_0(t))$. Therefore, since $\mathbb{X}_0$ is finite-dimensional, (u_0, v_0) is differentiable on $[0, +\infty)$ and solves (4.14) with the initial condition $(u(0), v(0)) = \mathbb{P}_0(\overline{u}, \overline{v})$. Similarly, since $\widetilde{\mathbb{P}}$ and $e^{t\mathbb{A}}$ commute, for $t \geq 0$, from the Duhamel formula we deduce that, for $t \geq 0$,

$$(\widetilde{u}(t), \widetilde{v}(t)) = e^{t\mathbb{A}}\widetilde{\mathbb{P}}(\overline{u}, \overline{v}) + \varepsilon \int_0^t e^{(t-\tau)\mathbb{A}}\, \widetilde{\mathbb{P}}\big(0, F_0(P_0 u(t))\big) = e^{t\mathbb{A}}\widetilde{\mathbb{P}}(\overline{u}, \overline{v}),$$

because $\{0\} \times X_0 \subset \operatorname{Ker} \widetilde{\mathbb{P}}$. Hence, we get that $(\widetilde{u}, \widetilde{v})$ is the mild solution of (4.15) with the initial condition $(u(0), v(0)) = \widetilde{\mathbb{P}}(\overline{u}, \overline{v})$, as desired.

We define the sets

$$\mathbb{U}_0 = U \times (B_R \cap X_0) \subset \mathbb{X}_0 \ \text{ and } \ \mathbb{U}_1 = \widetilde{B}'_r \times \widetilde{B}_R \subset \widetilde{\mathbb{X}}$$

where $\widetilde{B}_R = B_{\widetilde{X}}(0, R)$(2). Observe that

$$\mathbb{U}_0 \oplus \{(0,0)\} \subset \mathbb{U} \subset \mathbb{U}_0 \oplus \mathbb{U}_1. \tag{4.18}$$

(2)Recall that the space $\widetilde{X}$ is equipped with the norm $\|\cdot\|_{L^2}$.

The inclusion $\mathbb{U}_0 \oplus \{(0,0)\} \subset \mathbb{U}$ is clear. If $(u,v) \in \mathbb{U}$, then $u = u_0 + u_1$, where $u_0 \in U$ and $u_1 \in \widetilde{B}'_r$. On the other hand, $v = P_0 v + \widetilde{P} v$. Since projections P_0, $\widetilde{P}$ are orthogonal in $L^2(\mathbb{R}^N)$, $P_0 v \in B_R \cap X_0$ and $\widetilde{P} v \in \widetilde{B}_R$. Therefore, $(u_0, P_0 v) \in \mathbb{U}_0$ and $(u_1, \widetilde{P} v) \in \mathbb{U}_1$, which implies $\mathbb{U} \subset \mathbb{U}_0 \oplus \mathbb{U}_1$. By Lemma 4.1.2 there exists $\varepsilon_0 \in (0,1]$ such that

$$\Theta_T^{(\varepsilon)}(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0] \quad \text{and} \quad (\overline{u}, \overline{v}) \in \partial \mathbb{U}_0, \tag{4.19}$$

and

$$\operatorname{Ind}_B(\Theta_T^{(\varepsilon)}, \mathbb{U}_0) = (-1)^{\dim X_0} \deg_B(F_0, U) \;\; \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{4.20}$$

Moreover, from (4.19) we deduce that

$$\operatorname{Fix}(\Theta_T^{(\varepsilon)}, \overline{\mathbb{U}}_0) \subset \mathbb{U}_0 \;\; \text{for } \varepsilon \in (0, \varepsilon_0] \tag{4.21}$$

where

$$\operatorname{Fix}(\Theta_T^{(\varepsilon)}, \overline{\mathbb{U}}_0) = \{(u,v) \in \overline{\mathbb{U}}_0 : \Theta_T^{(\varepsilon)}(u,v) = (u,v)\}.$$

We claim that

$$\operatorname{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}) = \operatorname{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}_0 \oplus \mathbb{U}_1) \;\; \text{for } \varepsilon \in (0, \varepsilon_0]. \tag{4.22}$$

Indeed, since $\mathbb{U} \subset \mathbb{U}_0 \oplus \mathbb{U}_1$, in the light of the excision property for the topological index (see Theorem A.4.8), it is sufficient to show that

$$\operatorname{Fix}(\overline{\Phi}_T^{(\varepsilon)}, \overline{\mathbb{U}_0 \oplus \mathbb{U}_1}) \subset \mathbb{U} \;\; \text{for } \varepsilon \in (0, \varepsilon_0].$$

Let $(u,v) \in \operatorname{Fix}(\overline{\Phi}_T^{(\varepsilon)}, \overline{\mathbb{U}_0 \oplus \mathbb{U}_1})$. Since the projections $\mathbb{P}_0$ and $\widetilde{\mathbb{P}}$ are continuous, one has $\overline{\mathbb{U}_0 \oplus \mathbb{U}_1} = \overline{\mathbb{U}}_0 \oplus \overline{\mathbb{U}}_1$. Consequently, applying (4.16), $\mathbb{P}_0(u,v) \in \operatorname{Fix}(\Theta_T^{(\varepsilon)}, \overline{\mathbb{U}}_0)$ and $\widetilde{\mathbb{P}}(u,v) \in \operatorname{Fix}(\Gamma_T, \overline{\mathbb{U}}_1)$. In view of Theorem 2.4.4 (ii), $\widetilde{\mathbb{P}}(u,v) = (0,0)$. Hence, using (4.21) and (4.18), we get, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\operatorname{Fix}(\overline{\Phi}_T^{(\varepsilon)}, \overline{\mathbb{U}_0 \oplus \mathbb{U}_1}) \subset \operatorname{Fix}(\Theta_T^{(\varepsilon)}, \overline{\mathbb{U}}_0) \oplus \{(0,0)\} \subset \mathbb{U}_0 \oplus \{(0,0)\} \subset \mathbb{U}.$$

This proves (4.22). Now, taking into account (4.22) and the product formula for the topological index for k-set contractions – see Theorem A.4.9, we obtain, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\operatorname{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}) = \operatorname{Ind}_C(\Theta_T^{(\varepsilon)}, \mathbb{U}_0) \cdot \operatorname{Ind}_C(\Gamma_T, \mathbb{U}_1) = \operatorname{Ind}_B(\Theta_T^{(\varepsilon)}, \mathbb{U}_0) \cdot \operatorname{Ind}_C(\Gamma_T, \mathbb{U}_1).$$

By Theorem 2.4.4 (iii)

$$\operatorname{Ind}_C(\Gamma_T, \mathbb{U}_1) = (-1)^{m_-(-\Delta + \boldsymbol{V})}.$$

Taking together the two equations above and (4.20), we arrive finally at

$$\operatorname{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}) = (-1)^{m_-(-\Delta + \boldsymbol{V}) + \dim X_0} \deg_B(F_0, U) \;\; \text{for } \varepsilon \in (0, \varepsilon_0],$$

which completes the proof of the theorem. □

4.2 Resonant averaging principle

The theorem below presents a version of the resonant averaging principle and establishes an index formula for the evolution equations

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon\big(0, F(t, u(t))\big), \quad t > 0, \;\; \varepsilon \in (0,1], \tag{4.23}$$

under the assumption that $\varepsilon \in (0,1]$ is sufficiently small. Combined with a continuation argument, this result will enable us to compute the topological index for the nonlinear damped wave equation (4.1).

Theorem 4.2.1. *Let $\Phi_T^{(\varepsilon)} : \mathbb{X} \to \mathbb{X}$ be the translation operator associated with* (4.23) *and the mapping $\overline{F}_0 : X_0 \to X_0$ be given by*

$$\overline{F}_0(u) := \frac{1}{T}\int_0^T P_0 F(t,u)\,dt \quad \text{for } u \in X_0. \tag{4.24}$$

Suppose that r, $R > 0$ and $U \subset X_0$ is an open bounded set such that $\overline{F}_0(u) \neq 0$, for $u \in \partial U$, and let $\mathbb{U}$ be as in Theorem 4.1.1.

Then, there exists $\varepsilon_0 \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$\mathrm{Ind}_C(\Phi_T^{(\varepsilon)}, \mathbb{U}) = (-1)^{m_-(-\Delta+\boldsymbol{V})+\dim X_0} \deg_B(\overline{F}_0, U).$$

Remark 4.2.2. It is a version of the index formula obtained by Ćwiszewski [15, Thm. 3.1] and Kokocki [40, Thm. 4.1]. Both results pertain to equations on bounded domains. □

Before the proof we need the following auxiliary result.

Lemma 4.2.3. *Let $T_0 > 0$ and suppose that $(u,v) : [0,+\infty) \to \mathbb{X}$ is a T_0-periodic mild solution of*

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)), \quad t > 0. \tag{4.25}$$

Then, $u(t) = \varphi$, for all $t \geq 0$ and some $\varphi \in X_0$, and $v(t) = 0$, for all $t \geq 0$.

Proof. Since (u,v) is a T_0-periodic mild solution of (4.25), one has $(u(t), v(t)) = e^{t\mathbb{A}}(\varphi, \xi)$, for $t \geq 0$ and some $(\varphi, \xi) \in \mathbb{X}$, and $(\varphi, \xi) \in \mathrm{Ker}(e^{T_0\mathbb{A}} - I)$. In view of Lemma 2.4.1 (iv), $\mathrm{Ker}(e^{T_0\mathbb{A}} - I) = X_0 \times \{0\}$, thus $\varphi \in X_0$ and $\xi = 0$.

According to Proposition 2.3.6 (iii), $\mathbb{X}_0$ is invariant with respect to $e^{t\mathbb{A}}$, for all $t \geq 0$, hence, $(u(t), v(t)) \in \mathbb{X}_0$, for $t \geq 0$. The subspace $\mathbb{X}_0$ is finite-dimensional, thus (u,v) is differentiable on $[0,+\infty)$ and solves the system

$$\begin{cases} \dot{u}(t) = v(t), & t > 0 \\ \dot{v}(t) = -\beta v(t), & t > 0. \end{cases}$$

This implies that $v(t) = e^{-\beta t}\xi = 0$, for $t \geq 0$. Consequently, u is a constant function, which means that $u(t) = \varphi$, for $t \geq 0$, as desired. □

Proof of Theorem 4.2.1. Let $\Psi_T^{(\varepsilon)} : \mathbb{X} \times [0,1] \to \mathbb{X}$ be the translation operator associated with the evolution equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon\big(0, H(t, u(t), \mu)\big), \quad t > 0, \tag{4.26}$$

where $\varepsilon \in (0,1]$ is parameter and map $H : [0,+\infty) \times H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ is given by

$$H(t,u,\mu) = (1-\mu)F(t,u) + \mu\overline{F}_0(P_0 u).$$

Recall that $F : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous and satisfies conditions (**F1**) and (**F2**). By Proposition 3.2.1 the mapping F is completely continuous. Observe that, for any u_1 and $u_2 \in X_0$,

$$\begin{aligned} \|\overline{F}_0(u_1) - \overline{F}_0(u_2)\|_{L^2} &\leq \frac{1}{T}\int_0^T \|P_0F(t,u_1) - P_0F(t,u_2)\|_{L^2}\,dt \\ &\leq \frac{1}{T}\int_0^T L\|u_1 - u_2\|_{H^1}\,dt \leq LC\|u_1 - u_2\|_{L^2} \end{aligned}$$

where $L > 0$ is a constant from the condition (**F1**) and $C > 0$ is such that $\|u\|_{H^1} \leq C\|u\|_{L^2}$, for all $u \in X_0$, as all norms are equivalent in finite-dimensional spaces. Hence, $\overline{F}_0$ satisfies

(F1). Consequently, the mapping $\overline{F}_0 \circ P_0 : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ satisfies (**F1**). As X_0 is a finite-dimensional space, it is also completely continuous. Therefore, we infer that H is continuous, satisfies conditions (**H1**) and (**H2**) (see p. 54), and it is completely continuous. Thus, based on Theorem 3.4.1, for any $\varepsilon \in (0,1]$, the translation operator $\Psi_T^{(\varepsilon)} : \mathbb{X} \times [0,1] \to \mathbb{X}$ is well-defined and continuous, and it is a k-set contraction with respect to $\chi_{s,V}$. Note that, for any $\varepsilon \in (0,1]$, $\Psi_T^{(\varepsilon)}(\cdot,0) = \Phi_T^{(\varepsilon)}$ and $\Psi_T^{(\varepsilon)}(\cdot,1) = \overline{\Phi}_T^{(\varepsilon)}$, where $\overline{\Phi}_T^{(\varepsilon)}$ is the translation operator associated with

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon(0, \overline{F}_0(u(t))), \quad t > 0. \tag{4.27}$$

We claim that there exists $\varepsilon_0 \in (0,1]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\Psi_T^{(\varepsilon)}(\overline{u}, \overline{v}, \mu) \neq (\overline{u}, \overline{v}) \ \text{ for any } \ (\overline{u}, \overline{v}) \in \partial\mathbb{U} \text{ and } \mu \in [0,1]. \tag{4.28}$$

Indeed, suppose to the contrary that (4.28) does not hold, that is, there exist sequences $(\varepsilon_n)_{n\geq 1}$ in $(0,1]$, $(\mu_n)_{n\geq 1}$ in $[0,1]$ and $((\overline{u}_n, \overline{v}_n))_{n\geq 1}$ in $\partial\mathbb{U}$, such that $\varepsilon_n \to 0^+$ and

$$\Psi_T^{(\varepsilon_n)}(\overline{u}_n, \overline{v}_n, \mu_n) = (\overline{u}_n, \overline{v}_n) \ \text{ for } n \geq 1.$$

This means that, for any $n \geq 1$, there exists a T-periodic mild solution $(u_n, v_n) : [0, +\infty) \to \mathbb{X}$ of (4.26) with $\mu = \mu_n$, $\varepsilon = \varepsilon_n$ and $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$. Obviously, we may assume that $\mu_n \to \mu_0$ in $[0,1]$. On the other hand, define $\widetilde{\Psi}_T : \mathbb{X} \times [0,1]^2 \to \mathbb{X}$ as the translation operator associated with

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, \widetilde{H}(t, u(t), \mu, \varepsilon)\big), \quad t > 0,$$

where $\widetilde{H} : [0, +\infty) \times H^1(\mathbb{R}^N) \times [0,1]^2 \to L^2(\mathbb{R}^N)$ is defined by $\widetilde{H}(t, u, \mu, \varepsilon) = \varepsilon H(t, u, \mu)$. We readily see that $\widetilde{H}$ is continuous, satisfies (**H1**) and (**H2**), and it is completely continuous. Thus, in view of Theorem 3.4.1, the operator $\widetilde{\Psi}_T$ is a well-defined continuous k-set contraction with respect to $\chi_{s,V}$. Additionally, for any bounded subset $Z \subset \mathbb{X}$,

$$\chi_{s,V}(\Psi_T(Z \times [0,1]^2)) \leq e^{-\rho T} \chi_{s,V}(Z).$$

Observe that $\widetilde{\Psi}_T(\overline{u}, \overline{v}, \mu, \varepsilon) = \Psi_T^{(\varepsilon)}(\overline{u}, \overline{v}, \mu)$, for any $(\overline{u}, \overline{v}) \in \mathbb{X}$, $\mu \in [0,1]$ and $\varepsilon \in (0,1]$. Consequently, one has

$$\chi_{s,V}(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}) = \chi_{s,V}(\{\widetilde{\Psi}_T(\overline{u}_n, \overline{v}_n, \mu_n, \varepsilon_n)\}_{n\geq 1}) \leq e^{-\rho T} \chi_{s,V}(\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}),$$

i.e., the set $\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}$ is relatively compact and we may assume that $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$ in $\mathbb{X}$. Since $\partial\mathbb{U}$ is closed, we get $(\overline{u}_0, \overline{v}_0) \in \partial\mathbb{U}$. By the parameter continuity of mild solutions for evolution equations – see Theorem 3.4.1 (ii), $(u_n(t), v_n(t)) \to (u_0(t), v_0(t))$, uniformly on bounded subsets of $[0, +\infty)$ with respect to t, where $(u_0, v_0) : [0, +\infty) \to \mathbb{X}$ is a T-periodic mild solution of the problem

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)), \quad t > 0, \\ (u(0), v(0)) = (\overline{u}_0, \overline{v}_0). \end{cases}$$

Consequently, by Lemma 4.2.3, we have, for $t \geq 0$,

$$u_0(t) = \overline{u}_0 \in X_0 \ \text{ and } \ v_0(t) = 0. \tag{4.29}$$

Since $\overline{v}_0 = 0$ and

$$\partial\mathbb{U} = \partial(U \oplus \widetilde{B}_r') \times \overline{B_R} \ \cup \ \overline{U \oplus \widetilde{B}_r'} \times \partial B_R (^3),$$

(^{3})Recall that the norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{s,V}$ are equivalent (cf. Theorem 2.3.3). Hence, these norms generate the same topology and the boundary $\partial\mathbb{U}$ is taken in $\mathbb{X}$ with this topology. Moreover, the boundary $\partial(U \oplus \widetilde{B}_r')$ is in $H^1(\mathbb{R}^N)$ endowed with the topology generated by the norm $\|\cdot\|_{H^1}$ and the boundary ∂B_R is understood to be in $L^2(\mathbb{R}^N)$ with the topology generated by the norm $\|\cdot\|_{L^2}$.

we have $\overline{u}_0 \in \partial(U \oplus \widetilde{B}'_r)$ and, since $\overline{u}_0 \in X_0$ and

$$\partial(U \oplus \widetilde{B}'_r) = \partial U \oplus \overline{\widetilde{B}'_r} \cup \overline{U} \oplus \partial \widetilde{B}'_r(^4),$$

we obtain $\overline{u}_0 \in \partial U$.

We define $(p_n, q_n) : [0, +\infty) \to \mathbb{X}$, $n \geq 1$, as $(p_n(t), q_n(t)) = \mathbb{P}_0(u_n(t), v_n(t))$. In view of Proposition 2.3.6 (iii), $\mathbb{P}_0$ commutes with $e^{t\mathbb{A}}$, thus, by virtue of the Duhamel formula, (p_n, q_n), $n \geq 1$, are T-periodic mild solutions of

$$(\dot{p}(t), \dot{q}(t)) = \mathbb{A}(p(t), q(t)) + \varepsilon_n \mathbb{P}_0 \left(0, H(t, u_n(t), \mu_n)\right)$$

with the initial condition $\big(p(0), q(0)\big) = \mathbb{P}_0(\overline{u}_n, \overline{v}_n)$. The space $\mathbb{X}_0$ is finite-dimensional, thus, (p_n, q_n), $n \geq 1$, are differentiable on $[0, +\infty)$ and

$$\begin{cases} \dot{p}_n(t) = q_n(t), \\ \dot{q}_n(t) = -\beta q_n(t) + \varepsilon_n P_0 H(t, u_n(t), \mu_n), \end{cases}$$

which yields

$$\dot{q}_n(t) + \beta \dot{p}_n(t) = \varepsilon_n P_0 H(t, u_n(t), \mu_n). \tag{4.30}$$

After integrating (4.30) on $[0, T]$ and dividing by ε_n, one has

$$0 = \int_0^T P_0 H(t, u_n(t), \mu_n)\, dt. \tag{4.31}$$

Since $(u_n, v_n) \to (u_0, v_0)$ in $C([0, T], \mathbb{X})$ as $n \to \infty$ and H is continuous, we get that

$$P_0 H(\cdot, u_n(\cdot), \mu_n) \to P_0 H(\cdot, u_0(\cdot), \mu_0) \ \text{ in } C([0, T], L^2(\mathbb{R}^N)) \text{ as } n \to \infty.$$

Hence, we have

$$\int_0^T P_0 H(t, u_n(t), \mu_n)\, dt \to \int_0^T P_0 H(t, u_0(t), \mu_0)\, dt \ \text{ in } L^2(\mathbb{R}^N) \text{ as } n \to \infty.$$

Therefore, (4.31) yields

$$\int_0^T P_0 H(t, u_0(t), \mu_0)\, dt = 0.$$

From this and (4.29) we obtain

$$\begin{aligned} 0 &= \int_0^T P_0 H(t, \overline{u}_0, \mu_0)\, dt = \int_0^T P_0 \Big((1 - \mu_0) F(t, \overline{u}_0) + \mu_0 \overline{F}_0(\overline{u}_0) \Big)\, dt \\ &= (1 - \mu_0) \int_0^T P_0 F(t, \overline{u}_0)\, dt + \mu_0 T \overline{F}_0(\overline{u}_0) = T \overline{F}_0(\overline{u}_0). \end{aligned}$$

Hence, $\overline{F}_0(\overline{u}_0) = 0$, which contradicts the assumption that $\overline{F}_0(u) \neq 0$, for $u \in \partial U$.

The homotopy invariance of the index (see Theorem A.4.8) yields, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\mathrm{Ind}_C(\Phi_T^{(\varepsilon)}, \mathbb{U}) = \mathrm{Ind}_C(\Psi_T^{(\varepsilon)}(\cdot, 0), \mathbb{U}) = \mathrm{Ind}_C(\Psi_T^{(\varepsilon)}(\cdot, 1), \mathbb{U}) = \mathrm{Ind}_C(\overline{\Phi}_T^{(\varepsilon)}, \mathbb{U}).$$

Finally, decreasing $\varepsilon_0 > 0$ if necessary, we apply Theorem 4.1.1 to get the desired index formula. The proof of the theorem is completed. □

(^{4})The boundary ∂U is considered in X_0 with the topology generated by the norm $\|\cdot\|_{L^2}$ and the boundary $\partial \widetilde{B}'_r$ is taken in $\widetilde{X}'$ with the topology generated by the norm $\|\cdot\|_{H^1}$. Since all norms are equivalent in finite-dimensional spaces, the product topology on $H^1(\mathbb{R}^N)$ induced by the spaces X_0 and $\widetilde{X}'$ coincides with the topology generated by $\|\cdot\|_{H^1}$.

Using the resonant averaging principle we provide the following *continuation principle*.

Theorem 4.2.4. *Let $\overline{F}_0 : X_0 \to X_0$ be given by* (4.24)*, and let exist $R_\infty > 0$ such that*

(a) $\deg_B(\overline{F}_0, B_{R_\infty} \cap X_0) \neq 0$*, where $B_{R_\infty} = B_{L^2}(0, R_\infty)$;*

(b) *for $\varepsilon \in (0,1)$, equation* (4.23) *has no T-periodic mild solutions with $\|(u(0), v(0))\|_{\mathbb{X}} \geq R_\infty$.*

Then the equation (4.1) *admits a T-periodic mild solution.*

Proof. In view of Theorem 4.2.1, applied for $U = B_{R_\infty} \cap X_0$, $r = \|\widetilde{P}\|_{\mathcal{L}(H^1)} R_\infty$ and $R = R_\infty$, there exists $\varepsilon_0 \in (0,1]$ such that

$$\mathrm{Ind}_C(\Phi_T^{(\varepsilon)}, \mathbb{U}) = (-1)^{m_-(-\Delta+\boldsymbol{V})+\dim X_0} \deg_B(\overline{F}_0, U) \quad \text{for } \varepsilon \in (0, \varepsilon_0], \tag{4.32}$$

with $\mathbb{U} = U \oplus \widetilde{B}'_r \times B_{R_\infty}$. Let $H : [0,+\infty) \times H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ be defined by $H(t,u,\varepsilon) = \varepsilon F(t,u)$. We see that H is continuous, satisfies conditions (**H1**) and (**H2**), as well H is completely continuous. Thus, in view of Theorem 3.4.1, the translation along trajectories operator $\Psi_T : \mathbb{X} \times [0,1] \to \mathbb{X}$ associated with the equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, H(t, u(t), \varepsilon)\big), \quad t > 0,$$

is a well-defined continuous k-set contraction with respect to $\chi_{s,V}$. Moreover, $\Psi_T(u,v,\varepsilon) = \Phi_T^{(\varepsilon)}(u,v)$, for all $(u,v) \in \mathbb{X}$ and $\varepsilon \in (0,1]$, where $\Phi_T^{(\varepsilon)}$ is the translation operator associated with (4.23). Due to condition (b),

$$\Psi_T(u,v,\varepsilon) \neq (u,v), \quad \text{for all } \ (u,v) \in \mathbb{X} \setminus B_{\mathbb{X}}(0, R_\infty), \quad \varepsilon \in (0,1). \tag{4.33}$$

We claim that

$$B_{\mathbb{X}}(0, R_\infty) \subset \mathbb{U}. \tag{4.34}$$

In fact, let $(u,v) \in B_{\mathbb{X}}(0,R_\infty)$. Then $u \in B_{H^1}(0, R_\infty)$ and $v \in B_{R_\infty}$. Consequently, $\|P_0 u\|_{L^2} \leq \|u\|_{L^2} < R_\infty$, i.e. $P_0 u \in B_{R_\infty} \cap X_0$, and $\|\widetilde{P}u\|_{H^1} \leq \|\widetilde{P}\|_{\mathcal{L}(H^1)} \|u\|_{H^1} < \|\widetilde{P}\|_{\mathcal{L}(H^1)} R_\infty$, i.e. $\widetilde{P}u \in \widetilde{B}'_r$. Hence, $(u,v) \in \mathbb{U}$.

Therefore, either $\Phi_T^{(1)} = \Psi_T(\cdot, 1) : \mathbb{X} \to \mathbb{X}$ has a fixed point in $\partial\mathbb{U}$, which would imply the assertion, or, in view of (4.33) and (4.34), we have

$$\Psi_T(\overline{u}, \overline{v}, \varepsilon) \neq (\overline{u}, \overline{v}), \quad \text{for all } \ (\overline{u}, \overline{v}) \in \partial\mathbb{U}, \quad \varepsilon \in (0,1].$$

Applying the homotopy invariance of the topological index (see Theorem A.4.8), we obtain

$$\mathrm{Ind}_C(\Phi_T^{(1)}, \mathbb{U}) = \mathrm{Ind}_C(\Psi_T(\cdot,1), \mathbb{U}) = \mathrm{Ind}_C(\Psi_T(\cdot, \varepsilon_0), \mathbb{U}) = \mathrm{Ind}_C(\Phi_T^{(\varepsilon_0)}, \mathbb{U}).$$

This, together with (4.32) and the assumption (a), yields

$$\mathrm{Ind}_C(\Phi_T^{(1)}, \mathbb{U}) = (-1)^{m_-(-\Delta+\boldsymbol{V})+\dim X_0} \deg_B(\overline{F}_0, U) \neq 0.$$

Thus, by the existence property of the topological index (see Theorem A.4.8), there exists a T-periodic mild solution of (4.1), which ends the proof of the theorem. □

4.3 Geometric conditions for the nonlinear perturbation

We will work with the following *Landesman-Lazer conditions*:

$$(LL)_+ \qquad \int_0^T \int_{\{\varphi>0\}} \check{f}_+ \varphi \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} \hat{f}_- \varphi \, dx \, dt > 0 \quad \text{for any } \varphi \in \mathrm{Ker}\,(-\Delta + \boldsymbol{V}) \setminus \{0\}$$

and

$$(LL)_- \qquad \int_0^T \int_{\{\varphi>0\}} \widehat{f}_+\varphi \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} \check{f}_-\varphi \, dx \, dt < 0 \quad \text{for any } \varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$$

where

$$\check{f}_\pm(t,x) = \liminf_{s\to\pm\infty} f(t,x,s) \quad \text{and} \quad \widehat{f}_\pm(t,x) = \limsup_{s\to\pm\infty} f(t,x,s)$$

for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$.

Remark 4.3.1. Since function $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, it is measurable (cf. [2, Lemma 4.51]). Observe that, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$,

$$\check{f}_+(t,x) = \sup_{r\in\mathbb{R}} \inf_{s>r} f(t,x,s) = \sup_{n\in\mathbb{N}} \inf_{s>n} f(t,x,s).$$

Clearly, $(n,+\infty) \cap \mathbb{Q}$ is a dense subset of $(n,+\infty)$. Since $f(t,x,\cdot)$ is continuous, we see that $\{f(t,x,s) : s \in (n,+\infty) \cap \mathbb{Q}\}$ is dense in $\{f(t,x,s) : s \in (n,+\infty)\}$. Hence,

$$\check{f}_+(t,x) = \sup_{n\in\mathbb{N}} \inf_{s\in(n,+\infty)\cap\mathbb{Q}} f(t,x,s)$$

which means that $\check{f}_+ : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ is measurable. Similarly, functions $\check{f}_-$, $\widehat{f}_\pm$: $[0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ are measurable. In particular, functions $\check{f}_\pm(\cdot,x)$, $\widehat{f}_\pm(\cdot,x) : [0,+\infty) \to \mathbb{R}$ are measurable, for almost every $x \in \mathbb{R}^N$. As f is bounded by $m \in L^2(\mathbb{R}^N)$, one has $|\check{f}_\pm(t,x)|$, $|\widehat{f}_\pm(t,x)| \leq m(x)$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. This yields, for any $\varphi \in L^2(\mathbb{R}^N)$,

$$\int_0^T \int_{\mathbb{R}^N} |\check{f}_\pm\varphi| \, dx \, dt \leq T \int_{\mathbb{R}^N} |m\varphi| \, dx \leq \|m\|_{L^2} \|\varphi\|_{L^2} < +\infty$$

and the same estimate holds for $\widehat{f}_\pm$. Thus, based on the Fubini-Tonelli Theorem, $\check{f}_\pm\varphi$ and $\widehat{f}_\pm\varphi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ are integrable, and the integrals, both in $(LL)_+$ and $(LL)_-$, are well-defined and finite. □

Alternatively, we will work with the so-called *strong resonance conditions*:

$$(SR)_+ \qquad \begin{cases} \text{There exists } r \in L^2(\mathbb{R}^N) \text{ such that, for almost every } x \in \mathbb{R}^N, \\ r(x) \geq 0 \quad \text{and} \quad uf(t,x,u) \geq 0, \quad \text{for all } |u| \geq r(x) \text{ and all } t \in [0,T], \\ \displaystyle\int_0^T \int_{\{\varphi>0\}} k_+ \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} k_- \, dx \, dt > 0 \quad \text{for all } \varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\} \end{cases}$$

and

$$(SR)_- \qquad \begin{cases} \text{There exists } r \in L^2(\mathbb{R}^N) \text{ such that, for almost every } x \in \mathbb{R}^N, \\ r(x) \geq 0 \quad \text{and} \quad uf(t,x,u) \leq 0, \text{ for all } |u| \geq r(x) \text{ and all } t \in [0,T], \\ \displaystyle\int_0^T \int_{\{\varphi>0\}} k_+ \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} k_- \, dx \, dt < 0 \quad \text{for all } \varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\} \end{cases}$$

where the limits $k_\pm(t,x) = \lim_{u\to\pm\infty} uf(t,x,u)$ exist and are finite, for almost every $x \in \mathbb{R}^N$ and all $t \in [0,T]$.

Remark 4.3.2. Given that $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable (cf. Remark 4.3.1), functions $k_\pm : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ are also measurable. Moreover, if $(SR)_+$ is satisfied, then $k_\pm(t,x) \geq 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, thus, by Tonelli's Theorem, for any $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$ the integrals in the condition $(SR)_+$ are well-defined and non-negative. In particular, these integrals can attain $+\infty$. Similarly, if $(SR)_-$ is satisfied, then $k_\pm(t,x) \leq 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, thus, again by Tonelli's Theorem, the integrals in the condition $(SR)_-$ are well-defined and non-positive, in particular, they may equal $-\infty$. □

Remark 4.3.3. If either $(SR)_+$ or $(SR)_-$ holds, then both the limits $\lim_{u\to\pm\infty} uf(t,x,u)$ are finite, for almost every $x \in \mathbb{R}^N$ and all $t \in [0,T]$. Therefore $\lim_{u\to\pm\infty} f(t,x,u) = 0$, for almost every $x \in \mathbb{R}^N$ and all $t \in [0,T]$. □

Due to [27, Prop. 3 and Remark 2] we have the following version of the *unique continuation property.*

Theorem 4.3.4. [Unique continuation property] *Assume that $N \geq 3$ and $V = V_\infty + V_0$ is a Kato-Rellich type potential. If $u : \mathbb{R}^N \to \mathbb{R}$ is a nonzero weak solution of $-\Delta u + \boldsymbol{V} u = 0$, i.e., $u \in H^1_{loc}(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} \nabla u(x)\, \nabla\varphi(x) + V(x)\, u(x)\, \varphi(x)\, dx = 0 \quad \textit{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

then the set $\{u = 0\}$ has measure zero. In particular, if $u \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$, then the set $\{u = 0\}$ has measure zero.

The unique continuation property enables us to show that the conditions $(LL)_\pm$ or $(SR)_\pm$ are implied by conditions $(LL)'_\pm$ (see p. ix) or $(SR)'_\pm$ (see p. x), respectively, which can be verified without the explicit knowledge of $\operatorname{Ker}(-\Delta + \boldsymbol{V})$.

Lemma 4.3.5. *Let $N \geq 3$. If the condition $(LL)'_+$ is satisfied, then the Landesman-Lazer condition $(LL)_+$ is satisfied. Similarly, if the condition $(LL)'_-$ is fulfilled, then the Landesman-Lazer condition $(LL)_-$ is fulfilled.*

Remark 4.3.6. In view of Remark 4.3.1, functions $\check{f}_\pm$, $\hat{f}_\pm : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ are measurable and $|\check{f}_\pm(t,x)|$, $|\hat{f}_\pm(t,x)| \leq m(x)$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. Hence, $\check{f}_\pm(\cdot,x)$, $\hat{f}_\pm(\cdot,x) : [0,T] \to \mathbb{R}$ are integrable for almost every $x \in \mathbb{R}^N$, and consequently the functions $\check{h}_\pm, \hat{h}_\pm : \mathbb{R}^N \to \mathbb{R}$ are well-defined. □

Proof of Lemma 4.3.5. Suppose that $(LL)'_+$ is satisfied. Let $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$ and the set $E \subset \mathbb{R}^N$ be as in this condition. From Fubini's Theorem we deduce

$$\int_0^T \int_{\{\varphi>0\}} \check{f}_+ \varphi\, dx\, dt = \int_{\{\varphi>0\}} \varphi \int_0^T \check{f}_+\, dt\, dx = \int_{\{\varphi>0\}} \varphi\, \check{h}_+\, dx \geq 0$$

and

$$\int_0^T \int_{\{\varphi<0\}} \hat{f}_- \varphi\, dx\, dt = \int_{\{\varphi<0\}} \varphi \int_0^T \hat{f}_-\, dt\, dx = \int_{\{\varphi<0\}} \varphi\, \hat{h}_-\, dx \geq 0,$$

because $\check{h}_+(x) \geq 0$, $\hat{h}_-(x) \leq 0$, for almost every $x \in \mathbb{R}^N$. In view of the unique continuation property (see Theorem 4.3.4), $|\{\varphi = 0\}| = 0$, hence

$$|E| = |E \cap \{\varphi < 0\}| + |E \cap \{\varphi > 0\}|.$$

Observe that either $|E \cap \{\varphi > 0\}| > 0$ or $|E \cap \{\varphi < 0\}| > 0$, thus, because $\check{h}_+\Big|_E > 0$ and $\hat{h}_-\Big|_E < 0$, we obtain either

$$\int_{\{\varphi>0\}} \varphi\, \check{h}_+\, dx > 0 \quad \text{or} \quad \int_{\{\varphi<0\}} \varphi\, \hat{h}_-\, dx > 0.$$

This ends the proof in the case of $(LL)'_+$.

Assume now that $(LL)'_-$ holds, $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$ and the set $E \subset \mathbb{R}^N$ is as in the definition of $(LL)'_-$. Then, by Fubini's Theorem,

$$\int_0^T \int_{\{\varphi>0\}} \hat{f}_+ \varphi\, dx\, dt = \int_{\{\varphi>0\}} \varphi\, \hat{h}_+\, dx \leq 0$$

and the inequality is strict if $|E \cap \{\varphi > 0\}| > 0$, and

$$\int_0^T \int_{\{\varphi<0\}} \check{f}_- \varphi \, dx \, dt = \int_{\{\varphi<0\}} \varphi \, \check{h}_- \, dx \leq 0$$

and the inequality is strict if $|E \cap \{\varphi < 0\}| > 0$. Since, due to the unique continuation property, either $|E \cap \{\varphi > 0\}| > 0$ or $|E \cap \{\varphi < 0\}| > 0$, we get the desired inequality. The proof is completed. □

Lemma 4.3.7. *Let $N \geq 3$. Then the condition $(SR)'_+$ implies the strong resonance condition $(SR)_+$, and the condition $(SR)'_-$ implies the strong resonance condition $(SR)_-$.*

Remark 4.3.8. In the light of Remark 4.3.2, functions $k_\pm : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ are measurable. In particular, functions $k_\pm(\cdot, x) : [0, +\infty) \to \mathbb{R}$ are measurable, for almost every $x \in \mathbb{R}^N$. Furthermore, because $k_\pm(t, x) \geq 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, if $(SR)_+$ is fulfilled, and $k_\pm(t, x) \leq 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$, if $(SR)_-$ is satisfied, the integrals $\int_0^T k_\pm(t, x) \, dt$ are well-defined, for almost every $x \in \mathbb{R}^N$. □

Proof of Lemma 4.3.7. Assume that the condition $(SR)'_+$ is satisfied, $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$, and the set E is as in this condition. Taking into account Tonelli's Theorem, we obtain

$$\int_0^T \int_{\{\varphi>0\}} k_+ \, dx \, dt = \int_{\{\varphi>0\}} \int_0^T k_+ \, dt \, dx \geq 0 \text{ and } \int_0^T \int_{\{\varphi<0\}} k_- \, dx \, dt = \int_{\{\varphi<0\}} \int_0^T k_- \, dt \, dx \geq 0$$

because $k_\pm(t, x) \geq 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. By virtue of the unique continuation property (see Theorem 4.3.4), either $|E \cap \{\varphi > 0\}| > 0$ or $|E \cap \{\varphi < 0\}| > 0$, thus, since $\int_0^T k_\pm \, dt > 0$ for $x \in E$, we have either

$$\int_{\{\varphi>0\}} \int_0^T k_+ \, dt \, dx > 0 \text{ or } \int_{\{\varphi<0\}} \int_0^T k_- \, dt \, dx > 0.$$

This ends the proof in the case of the strong resonance type condition $(SR)'_+$.

Now, assume that the condition $(SR)'_-$ is fulfilled, $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$ and E is like in this condition. As before, in view of Tonelli's Theorem,

$$\int_0^T \int_{\{\varphi>0\}} k_+ \, dx \, dt \leq 0 \text{ and } \int_0^T \int_{\{\varphi<0\}} k_- \, dx \, dt \leq 0$$

and the first inequality is strict if $|E \cap \{\varphi > 0\}| > 0$, and the second inequality is strict if $|E \cap \{\varphi < 0\}| > 0$. Consequently, by the unique continuation property, either $|E \cap \{\varphi > 0\}| > 0$ or $|E \cap \{\varphi < 0\}| > 0$, which completes the proof. □

The next two results will establish connections between the properties of f and the geometry of the Nemytskii operator F.

Lemma 4.3.9. *Let $(\mu_n)_{n\geq1}$ in $(0, +\infty)$ and $(u_n)_{n\geq1}$ in $C([0, T], H^1(\mathbb{R}^N))$ are such that $\mu_n \to +\infty$ and $u_n \to \varphi$ in $C([0, T], H^1(\mathbb{R}^N))$ as $n \to \infty$, for some $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$. Then*

$$\liminf_{n\to\infty} \int_0^T \langle F(t, \mu_n u_n(t)), P_0 u_n(t) \rangle_{L^2} \, dt \geq \int_0^T \int_{\{\varphi>0\}} \check{f}_+ \varphi \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} \hat{f}_- \varphi \, dx \, dt \tag{4.35}$$

and

$$\limsup_{n\to\infty} \int_0^T \langle F(t, \mu_n u_n(t)), P_0 u_n(t) \rangle_{L^2} \, dt \leq \int_0^T \int_{\{\varphi>0\}} \hat{f}_+ \varphi \, dx \, dt + \int_0^T \int_{\{\varphi<0\}} \check{f}_- \varphi \, dx \, dt. \tag{4.36}$$

Proof. Since sequence $(u_n)_{n\geq 1}$ is convergent in $C([0,T], H^1(\mathbb{R}^N))$ and f is bounded by a function $m \in L^2(\mathbb{R}^N)$, there exists a constant $C > 0$ such that

$$\left|\langle F(t, \mu_n u_n(t)), P_0 u_n(t)\rangle_{L^2}\right| \leq C \quad \text{for all } n \geq 1 \text{ and } t \in [0,T].$$

This allows us to apply Fatou's Lemma:

$$\liminf_{n\to\infty} \int_0^T \langle F(t, \mu_n u_n(t)), P_0 u_n(t)\rangle_{L^2}\, dt \geq \int_0^T \liminf_{n\to\infty} \langle F(t, \mu_n u_n(t)), P_0 u_n(t)\rangle_{L^2}\, dt. \tag{4.37}$$

Fix $t \in [0,T]$. Then there exists a sequence of positive integers $(n_k)_{k\geq 1}$ such that

$$\liminf_{n\to\infty} \langle F(t, \mu_n u_n(t)), P_0 u_n(t)\rangle_{L^2} = \lim_{k\to\infty} \langle F(t, \mu_{n_k} u_{n_k}(t)), P_0 u_{n_k}(t)\rangle_{L^2}. \tag{4.38}$$

Next, we choose a further subsequence (for simplicity denoted by $(u_{n_k})_{k\geq 1}$) such that $u_{n_k}(t)(x) \to \varphi(x)$ and $P_0 u_{n_k}(t)(x) \to \varphi(x)$, for almost every $x \in \mathbb{R}^N$. In addition, there is $h \in L^2(\mathbb{R}^N)$ such that $|P_0 u_{n_k}(t)(x)| \leq h(x)$, for all $k \geq 1$ and almost every $x \in \mathbb{R}^N$ – see Proposition A.2.2 (ii).

Observe that, for all $k \geq 1$ and almost every $x \in \mathbb{R}^N$,

$$\left|f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\right| \leq m(x)h(x).$$

Hence, applying again Fatou's Lemma, we have

$$\begin{aligned}
&\lim_{k\to\infty} \big\langle F\big(t, \mu_{n_k} u_{n_k}(t), P_0 u_{n_k}(t)\big)\big\rangle_{L^2} = \lim_{k\to\infty} \int_{\mathbb{R}^N} f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\, dx \\
&\geq \int_{\mathbb{R}^N} \liminf_{k\to\infty} \Big(f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\Big)\, dx \\
&= \int_{\{\varphi>0\}} \liminf_{k\to\infty} \Big(f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\Big)\, dx \\
&\quad + \int_{\{\varphi=0\}} \liminf_{k\to\infty} \Big(f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\Big)\, dx \\
&\quad + \int_{\{\varphi<0\}} \liminf_{k\to\infty} \Big(f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, P_0 u_{n_k}(t)(x)\Big)\, dx \\
&= \int_{\{\varphi>0\}} \liminf_{k\to\infty} f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, \varphi(x)\, dx + \int_{\{\varphi<0\}} \limsup_{k\to\infty} f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big)\, \varphi(x)\, dx.
\end{aligned}$$

Taking into account that, for almost every $x \in \mathbb{R}^N$,

$$\liminf_{k\to\infty} f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big) \geq \liminf_{r\to+\infty} f(t, x, r) = \check{f}_+(t, x)$$

and

$$\limsup_{k\to\infty} f\big(t, x, \mu_{n_k} u_{n_k}(t)(x)\big) \leq \limsup_{r\to-\infty} f(t, x, r) = \hat{f}_-(t, x)$$

we arrive at

$$\lim_{k\to\infty} \langle F(t, \mu_{n_k} u_{n_k}(t)), P_0 u_{n_k}(t)\rangle_{L^2} \geq \int_{\{\varphi>0\}} \check{f}_+ \varphi\, dx + \int_{\{\varphi<0\}} \hat{f}_- \varphi\, dx.$$

Using the inequality above, (4.38) and (4.37), we get (4.35).

In order to obtain (4.36) we apply the analogical arguments. □

Lemma 4.3.10. *Let $N \geq 3$. Suppose that $(\mu_n)_{n\geq 1}$ is a sequence in $(0, +\infty)$ such that $\mu_n \to +\infty$ and $(u_n)_{n\geq 1}$ is a sequence in $C([0,T], H^1(\mathbb{R}^N))$ such that $u_n \to \varphi$ in $C([0,T], H^1(\mathbb{R}^N))$ as $n \to \infty$ with $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V}) \setminus \{0\}$. If there exists $r \in L^2(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$, $r(x) \geq 0$ and*

$$uf(t, x, u) \geq 0 \quad \textit{for all } \ |u| \geq r(x) \ \textit{ and all } \ t \in [0,T], \tag{4.39}$$

then

$$\liminf_{n\to\infty}\int_0^T \langle F(t,\mu_n u_n(t)),\mu_n u_n(t)\rangle_{L^2}\,dt \geq \int_0^T\int_{\{\varphi>0\}} k_+\,dx\,dt + \int_0^T\int_{\{\varphi<0\}} k_-\,dx\,dt. \tag{4.40}$$

Moreover, if there exists $r\in L^2(\mathbb{R}^N)$ *such that, for almost every* $x\in\mathbb{R}^N$, $r(x)\geq 0$ *and*

$$uf(t,x,u)\leq 0 \quad \textit{for all}\ \ |u|\geq r(x) \quad \textit{and all}\ \ t\in[0,T], \tag{4.41}$$

then

$$\limsup_{n\to\infty}\int_0^T \langle F(t,\mu_n u_n(t)),\mu_n u_n(t)\rangle_{L^2}\,dt \leq \int_0^T\int_{\{\varphi>0\}} k_+\,dx\,dt + \int_0^T\int_{\{\varphi<0\}} k_-\,dx\,dt. \tag{4.42}$$

Proof. We will deal only with (4.40), because the inequality (4.42) can be proved analogously. For each $n\geq 1$ and $t\in[0,T]$, let us denote

$$\widetilde{u}_n(t)=\mu_n u_n(t), \quad B_n=\{x\in\mathbb{R}^N : |\widetilde{u}_n(t)(x)|<r(x)\}.$$

Observe that, for all $n\geq 1$ and all $t\in[0,T]$,

$$\begin{aligned}\big\langle F\big(t,\widetilde{u}_n(t)\big),\widetilde{u}_n(t)\big\rangle_{L^2} &= \big\langle F\big(t,\widetilde{u}_n(t)\big),\widetilde{u}_n(t)\mathbf{1}_{B_n}\big\rangle_{L^2} + \big\langle F\big(t,\widetilde{u}_n(t)\big),\widetilde{u}_n(t)\mathbf{1}_{\mathbb{R}^N\setminus B_n}\big\rangle_{L^2}\\ &= \big\langle F\big(t,\widetilde{u}_n(t)\mathbf{1}_{B_n}\big),\widetilde{u}_n(t)\mathbf{1}_{B_n}\big\rangle_{L^2} + \big\langle F\big(t,\widetilde{u}_n(t)\mathbf{1}_{\mathbb{R}^N\setminus B_n}\big),\widetilde{u}_n(t)\mathbf{1}_{\mathbb{R}^N\setminus B_n}\big\rangle_{L^2}\\ &= F_n(t)+G_n(t)\end{aligned} \tag{4.43}$$

where

$$F_n(t)=\big\langle F\big(t,\widetilde{u}_n(t)\mathbf{1}_{B_n}\big),\widetilde{u}_n(t)\mathbf{1}_{B_n}\big\rangle_{L^2}, \quad G_n(t)=\big\langle F\big(t,\widetilde{u}_n(t)\mathbf{1}_{\mathbb{R}^N\setminus B_n}\big),\widetilde{u}_n(t)\mathbf{1}_{\mathbb{R}^N\setminus B_n}\big\rangle_{L^2}.$$

From inequality (4.39) we deduce that, for all $n\geq 1$, all $t\in[0,T]$, and almost every $x\in\mathbb{R}^N$,

$$f\big(t,x,\widetilde{u}_n(t)(x)\,\mathbf{1}_{B_n}(x)\big)\,\widetilde{u}_n(t)(x)\,\mathbf{1}_{B_n}(x)\geq 0,$$

which yields $F_n(t)\geq 0$, for all $n\geq 1$ and all $t\in[0,T]$. Therefore, by the Fatou Lemma,

$$\liminf_{n\to\infty}\int_0^T F_n(t)\,dt \geq \int_0^T \liminf_{n\to\infty} F_n(t)\,dt. \tag{4.44}$$

We prove that

$$G_n(t)\xrightarrow{n\to\infty} 0 \quad \text{for all } t\in[0,T]. \tag{4.45}$$

To this end, we choose an arbitrary subsequence $(G_{n_k}(t))_{k\geq 1}$ and we show that $(G_{n_k}(t))_{k\geq 1}$ contains a subsequence that converges to 0. By Proposition A.2.2 (ii), passing to a subsequence if necessary,

$$u_{n_k}(t)(x)\xrightarrow{k\to\infty}\varphi(x) \quad \text{for all } x\in B$$

where $B\subset\mathbb{R}^N$ is such that $|\mathbb{R}^N\setminus B|=0$. We prove now that

$$\mathbf{1}_{\mathbb{R}^N\setminus B_{n_k}}(x)\xrightarrow{k\to\infty} 0 \quad \text{for all } x\in\{\varphi\neq 0\}\cap B. \tag{4.46}$$

Indeed, let $x\in\{\varphi\neq 0\}\cap B$. There exists $k_1\geq 1$ such that $|u_{n_k}(t)(x)|\geq\frac{1}{2}|\varphi(x)|$ for all $k\geq k_1$. Hence there exists $k_2\geq k_1$ such that $|\widetilde{u}_{n_k}(t)(x)|\geq r(x)$ for all $k\geq k_2$, i.e.,

$$x\notin\mathbb{R}^N\setminus B_{n_k} \quad \text{for all } k\geq k_2,$$

which proves (4.46). It follows from the unique continuation property (see Theorem 4.3.4) that

$$|\mathbb{R}^N \setminus (\{\varphi \neq 0\} \cap B)| = 0,$$

hence (4.46) is satisfied for almost every $x \in \mathbb{R}^N$. Moreover, observe that, for all $k \geq 1$, all $t \in [0,T]$, and almost every $x \in \mathbb{R}^N$,

$$\left| f\big(t, x, \widetilde{u}_n(t)(x)\, \mathbf{1}_{\mathbb{R}^N \setminus B_n}(x)\big)\, \widetilde{u}_n(t)(x)\, \mathbf{1}_{\mathbb{R}^N \setminus B_n}(x) \right| \leq m(x) r(x). \tag{4.47}$$

Since $m(\cdot)r(\cdot) \in L^1(\mathbb{R}^N)$, it follows from the Lebesgue Dominated Convergence Theorem that $G_{n_k}(t) \xrightarrow{k\to\infty} 0$. Since a subsequence $(G_{n_k}(t))_{k\geq 1}$ is arbitrary, this proves (4.45). By (4.47),

$$|G_n(t)| \leq \|m\|_{L^2}\, \|r\|_{L^2} \quad \text{for all } t \in [0,T].$$

Therefore, using the Dominated Convergence Theorem once again, we obtain

$$\lim_{n\to\infty} \int_0^T G_n(t)\, dt = \int_0^T \lim_{n\to\infty} G_n(t)\, dt = 0. \tag{4.48}$$

Combining (4.43) and (4.44) with (4.48), we arrive at

$$\begin{aligned} \liminf_{n\to\infty} \int_0^T \big\langle F\big(t, \widetilde{u}_n(t)\big), \widetilde{u}_n(t) \big\rangle_{L^2} &= \liminf_{n\to\infty} \int_0^T F_n(t)\, dt + \int_0^T G_n(t)\, dt \\ &\geq \int_0^T \liminf_{n\to\infty} F_n(t)\, dt. \end{aligned} \tag{4.49}$$

Now, we set some $t \in [0,T]$ and extract a subsequence $(F_{n_k})_{k\geq 1}$ such that

$$\liminf_{n\to\infty} F_n(t) = \lim_{k\to\infty} F_{n_k}(t). \tag{4.50}$$

Further, we can assume, passing to a subsequence if necessary, that $u_{n_k}(t)(x) \xrightarrow{k\to\infty} \varphi(x)$, for almost every $x \in \mathbb{R}^N$ (cf. Proposition A.2.2 (ii)). Therefore,

$$\widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x) \xrightarrow{k\to\infty} +\infty \quad \text{for almost every } x \in \{\varphi > 0\}$$

and

$$\widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x) \xrightarrow{k\to\infty} -\infty \quad \text{for almost every } x \in \{\varphi < 0\}.$$

Then, the Fatou Lemma and the unique continuation property yield

$$\begin{aligned} \lim_{k\to\infty} F_{n_k}(t) &\geq \int_{\mathbb{R}^N} \liminf_{k\to\infty} f\big(t, x, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\big)\, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\, dx \\ &= \int_{\{\varphi>0\}} \liminf_{k\to\infty} f\big(t, x, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\big)\, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\, dx \\ &\quad + \int_{\{\varphi<0\}} \liminf_{k\to\infty} f\big(t, x, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\big)\, \widetilde{u}_{n_k}(t)(x)\, \mathbf{1}_{B_{n_k}}(x)\, dx \\ &= \int_{\{\varphi>0\}} k_+(t,x)\, dx + \int_{\{\varphi<0\}} k_-(t,x)\, dx. \end{aligned}$$

Since $t \in [0,T]$ is arbitrary, from (4.50) we deduce that

$$\liminf_{n\to\infty} F_n(t) \geq \int_{\{\varphi>0\}} k_+(t,x)\, dx + \int_{\{\varphi<0\}} k_-(t,x)\, dx \quad \text{for all } t \in [0,T].$$

Combining this with (4.49) we obtain (4.40). $\square$

Lemma 4.3.11. *Suppose that* $(\mu_n)_{n\geq 1}$ *is a sequence in* $(0,+\infty)$ *such that* $\mu_n \to +\infty$ *and* $(u_n)_{n\geq 1}$ *is a sequence in* $C([0,T],H^1(\mathbb{R}^N))$ *such that* $u_n \to \varphi$ *in* $C([0,T],H^1(\mathbb{R}^N))$ *as* $n\to\infty$, *for some* $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V})\setminus\{0\}$, *and* $(w_n)_{n\geq 1}$ *is a bounded sequence in* $C([0,T],H^1(\mathbb{R}^N))$. *If* $\lim_{u\to\pm\infty} f(t,x,u) = 0$, *for almost every* $x\in\mathbb{R}^N$ *and all* $t\in[0,T]$, *then*

$$\lim_{n\to\infty}\int_0^T \langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2}\,dt = 0.$$

Proof. As f is bounded by $m\in L^2(\mathbb{R}^N)$, we have, for any $t\in[0,T]$ and $n\geq 1$,

$$|\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2}| \leq \|m\|_{L^2}\,C$$

where $C>0$ is a bound of the sequence $(w_n)_{n\geq 1}$. Therefore, in view of the Lebesgue Dominated Convergence Theorem, it suffices to prove that, for any fixed $t\in[0,T]$,

$$\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2} \to 0 \quad \text{as } n\to\infty.$$

To this end, take any $\varepsilon>0$ and let $R>0$ be such that $\|m\|_{L^2(\mathbb{R}^N\setminus B(0,R))} < \varepsilon/C$. Thus, for all $n\geq 1$ and $t\in[0,T]$,

$$\begin{aligned}|\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2}| &\leq |\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2(B(0,R))}| + |\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2(\mathbb{R}^N\setminus B(0,R))}|\\ &< |\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2(B(0,R))}| + \varepsilon.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}|\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2}| \leq \limsup_{n\to\infty}|\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2(B(0,R))}| + \varepsilon.$$

Let $(n_k)_{k\geq 1}$ be such that

$$\limsup_{n\to\infty}|\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2(B(0,R))}| = \lim_{k\to\infty}|\langle F(t,\mu_{n_k} u_{n_k}(t)), w_{n_k}(t)\rangle_{L^2(B(0,R))}|.$$

Now, by the Rellich-Kondrachov Theorem (see Theorem A.2.5), we may assume that

$$w_{n_k}(t)\to w \ \text{ in } L^2(B(0,R)) \text{ as } k\to\infty.$$

Consequently, extracting a subsequence if necessary, $u_{n_k}(t)(x)\to\varphi(x)$ and $w_{n_k}(t)(x)\to w(x)$, for almost every $x\in B(0,R)$, and there exists $h\in L^2(B(0,R))$ such that $|w_{n_k}(t)(x)|\leq h(x)$, for almost every $x\in B(0,R)$ (cf. Proposition A.2.2 (ii)). Therefore, using the Lebesgue Dominated Convergence Theorem and the unique continuation property (see Theorem 4.3.4), we have

$$\begin{aligned}&\lim_{k\to\infty}\int_{B(0,R)} f\big(t,x,\mu_{n_k}u_{n_k}(t)(x)\big)\,w_{n_k}(t)(x)\,dx\\ &=\int_{B(0,R)}\lim_{k\to\infty} f\big(t,x,\mu_{n_k}u_{n_k}(t)(x)\big)\,w_{n_k}(t)(x)\,dx\\ &=\int_{B(0,R)\cap\{\varphi>0\}}\lim_{k\to\infty}\Big(f\big(t,x,\mu_{n_k}u_{n_k}(t)(x)\big)\,w_{n_k}(t)(x)\Big)\,dx\\ &\quad+\int_{B(0,R)\cap\{\varphi<0\}}\lim_{k\to\infty}\Big(f\big(t,x,\mu_{n_k}u_{n_k}(t)(x)\big)\,w_{n_k}(t)(x)\Big)\,dx.\end{aligned} \tag{4.51}$$

Note that, for almost every $x\in B(0,R)\cap\{\varphi>0\}$, $u_{n_k}(t)(x)\to\varphi(x)>0$ as $k\to\infty$. Hence $\mu_{n_k}u_{n_k}(t)(x)\xrightarrow{k\to\infty}+\infty$ and we obtain

$$\lim_{k\to\infty} f(t,x,\mu_{n_k}u_{n_k}(t)(x)) = \lim_{u\to+\infty} f(t,x,u) = 0 \quad \text{for almost every } x\in B(0,R)\cap\{\varphi>0\}.$$

Similarly, we get

$$\lim_{k\to\infty} f(t,x,\mu_{n_k}u_{n_k}(t)(x)) = \lim_{u\to-\infty} f(t,x,u) = 0 \quad \text{for almost every } x\in B(0,R)\cap\{\varphi<0\}.$$

Hence,

$$\limsup_{n\to\infty} |\langle F(t,\mu_n u_n(t)), w_n(t)\rangle_{L^2}| \le \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, the proof is completed. □

The next result provides well-known index formulae on the finite-dimensional space $X_0 = \mathrm{Ker}(-\Delta+\boldsymbol{V})$ under Landesman-Lazer or strong resonance conditions.

Lemma 4.3.12. *If one of the conditions* $(LL)_\pm$ *or* $(SR)_\pm$ *is satisfied, and* $N\ge 3$ *in the case of* $(SR)_\pm$*, then there exists* $R_0>0$ *such that*

$$\pm\langle \overline{F}_0(u),u\rangle_{L^2} > 0, \quad \textit{for all} \quad u\in X_0\setminus B_{L^2}(0,R_0) \tag{4.52}$$

where $\overline{F}_0 : X_0\to X_0$ *is given by* (4.24). *Moreover, for any* $R\ge R_0$,

$$\deg_B(\overline{F}_0, B_{L^2}(0,R)\cap X_0) = \begin{cases} 1, & \textit{if } (LL)_+ \textit{ or } (SR)_+ \textit{ holds,}\\ (-1)^{\dim X_0}, & \textit{if } (LL)_- \textit{ or } (SR)_- \textit{ holds.}\end{cases} \tag{4.53}$$

Proof. We shall show (4.52) under conditions $(SR)_-$ or $(LL)_-$. With conditions $(SR)_+$ or $(LL)_+$ one can proceed analogously. Suppose to the contrary that (4.52) is not satisfied, which implies the existence of a sequence $(u_n)_{n\ge1}$ in X_0 such that $\|u_n\|_{L^2}\to+\infty$ and

$$\langle \overline{F}_0(u_n),u_n\rangle_{L^2}\ge 0 \quad \text{for all } n\ge 1.$$

Consequently,

$$\begin{aligned}\langle \overline{F}_0(u_n),u_n\rangle_{L^2} &= \Big\langle \frac{1}{T}\int_0^T P_0F(t,u_n)\,dt, u_n\Big\rangle_{L^2}\\ &= \Big\langle \frac{1}{T}\int_0^T F(t,u_n)\,dt, u_n\Big\rangle_{L^2} = \frac{1}{T}\int_0^T \langle F(t,u_n),u_n\rangle_{L^2}\,dt\end{aligned}$$

and

$$\int_0^T \langle F(t,u_n),u_n\rangle_{L^2}\,dt \ge 0 \quad \text{for all } n\ge 1. \tag{4.54}$$

Subsequently, we put $\mu_n = \|u_n\|_{L^2}$ and $\widetilde{u}_n = u_n/\mu_n$. Then $(\widetilde{u}_n)_{n\ge1}$ is a bounded sequence in the finite-dimensional space X_0, so we may assume that $\widetilde{u}_n\to\varphi$ in $H^1(\mathbb{R}^N)$, for some $\varphi\in X_0\setminus\{0\}$.

If $(SR)_-$ is satisfied, then, in the light of Lemma 4.3.10,

$$\limsup_{n\to\infty}\int_0^T \langle F(t,u_n),u_n\rangle_{L^2}\,dt = \limsup_{n\to\infty}\int_0^T \langle F(t,\mu_n\widetilde{u}_n),\mu_n\widetilde{u}_n\rangle_{L^2}\,dt < 0,$$

a contradiction with (4.54). Observe that, if $(LL)_-$ holds, then, in view of Lemma 4.3.9,

$$\limsup_{n\to\infty}\int_0^T \langle F(t,\mu_n\widetilde{u}_n),\widetilde{u}_n\rangle_{L^2}\,dt < 0,$$

again a contradiction with (4.54).

In order to show (4.53), we assume firstly that one of the conditions $(LL)_-$ or $(SR)_-$ is satisfied. Let us define the map $H : X_0\times[0,1]\to X_0$ as follows

$$H(u,\mu) = (1-\mu)\overline{F}_0(u) - \mu u.$$

Obviously, H is continuous. Next, let $\mu \in [0,1]$ and $u \in X_0$ be such that $\|u\|_{L^2} = R$, where $R \geq R_0$ and $R_0 > 0$ is as in (4.52). Hence, (4.52) yields

$$\langle H(u,\mu), u\rangle_{L^2} = (1-\mu)\langle \overline{F}_0(u), u\rangle_{L^2} - \mu\|u\|^2_{L^2} < 0,$$

which implies $H(u,\mu) \neq 0$. Therefore, it follows from the homotopy invariance that

$$\begin{aligned}\deg_B(\overline{F}_0, B_{L^2}(0,R)\cap X_0) &= \deg_B(H(\cdot,0), B_{L^2}(0,R)\cap X_0) = \deg_B(H(\cdot,1), B_{L^2}(0,R)\cap X_0)\\ &= \deg_B(-I, B_{L^2}(0,R)\cap X_0) = (-1)^{\dim X_0}.\end{aligned}$$

In the case of the conditions $(LL)_+$ or $(SR)_+$ we consider the homotopy $H : X_0 \times [0,1] \to X_0$ given by

$$H(u,\mu) = (1-\mu)\overline{F}_0(u) + \mu u.$$

Therefore, the proof is completed. □

4.4 Proof of Theorem I

In this section we will prove a slightly extended version of Theorem I involving Landesman-Lazer $(LL)_\pm$ and strong resonance $(SR)_\pm$ conditions, which are more general than the conditions $(LL)'_\pm$ and $(SR)'_\pm$, respectively; see Lemma 4.3.5 and Lemma 4.3.7.

Theorem 4.4.1. *Assume that $V : \mathbb{R}^N \to \mathbb{R}$, where $N \geq 1$, is a Kato-Rellich type potential such that the asymptotic bottom of the $L^\infty(\mathbb{R}^N)$-part of V is positive, that is, conditions* (5a), (5b)*, and* (7) *are satisfied. Suppose that the nonlinear term f is a T-periodic Carathéodory function (see conditions (P) and (C) on p. ix) satisfying the Lipschitz condition* (f1) *(see p. ix). Moreover, assume that the nonlinear damped wave equation* (4.1) *is at resonance at infinity, i.e.,* $\mathrm{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\}$ *and f satisfies condition* (f2)$'$ *(see p. ix).*

(i) If one of the Landesman-Lazer conditions $(LL)_\pm$ is satisfied, then equation (4.1) *admits a T-periodic mild solution. Moreover, the assertion is also true if $N \geq 3$ and one of the Landesman-Lazer type conditions $(LL)'_\pm$ holds.*

(ii) If $N \geq 3$ and one of the strong resonance conditions $(SR)_\pm$ is satisfied, then equation (4.1) *admits a T-periodic mild solution. Furthermore, the assertion is also true if one of the strong resonance type conditions $(SR)'_\pm$ is satisfied.*

To complete the proof of Theorem 4.4.1 we will need the following two lemmata.

Lemma 4.4.2. *Assume that sequence of continuous functions $\big(w_n : [0,+\infty) \to \mathbb{X}\big)_{n\geq 1}$ is such that $w_n \to 0$ in $C([0,T],\mathbb{X})$ as $n \to \infty$, $\big((\overline{u}_n, \overline{v}_n)\big)_{n\geq 1}$ is bounded in $\mathbb{X}$ and $(u_n, v_n) : [0,+\infty) \to \mathbb{X}$, $n \geq 1$, are T-periodic mild solutions of*

$$\begin{cases}(\dot u(t), \dot v(t)) = \mathbb{A}(u(t), v(t)) + w_n(t), & t > 0,\\ (u(0), v(0)) = (\overline{u}_n, \overline{v}_n).\end{cases}$$

Then there exists a subsequence $\big((u_{n_k}, v_{n_k})\big)_{k\geq 1}$ such that $(u_{n_k}, v_{n_k}) \to (\varphi, 0)$ in $C([0,T],\mathbb{X})$, for some $\varphi \in \mathrm{Ker}\,(-\Delta + \boldsymbol{V})$.

Proof. Based on periodicity of (u_n, v_n), $n \geq 1$, the Duhamel formula (see (1.5) in Section 1.5) and the algebraic semi-additivity of the Hausdorff measure of non-compactness $\chi_{s,V}$, we obtain

$$\begin{aligned}\chi_{s,V}\Big(\big\{(\overline{u}_n, \overline{v}_n)\big\}_{n\geq 1}\Big) &= \chi_{s,V}\Big(\big\{(u_n(T), v_n(T)\big\}_{n\geq 1}\Big)\\ &\leq \chi_{s,V}\Big(\big\{e^{T\mathbb{A}}(\overline{u}_n, \overline{v}_n)\big\}_{n\geq 1}\Big) + \chi_{s,V}\left(\left\{\int_0^T e^{(T-t)\mathbb{A}} w_n(t)\,dt\right\}_{n\geq 1}\right).\end{aligned}$$

By Theorem 2.3.9:

$$\chi_{s,V}\Big(\big\{e^{T\mathbb{A}}(\overline{u}_n,\overline{v}_n)\big\}_{n\geq 1}\Big) \leq e^{-\rho T}\chi_{s,V}\Big(\big\{(\overline{u}_n,\overline{v}_n)\big\}_{n\geq 1}\Big).$$

Since, by assumption, $w_n \to 0$ in $C([0,T],\mathbb{X})$, we have

$$\lim_{n\to\infty}\int_0^T e^{(T-t)\mathbb{A}}w_n(t)\,dt = 0.$$

Therefore,

$$\chi_{s,V}\Big(\big\{(\overline{u}_n,\overline{v}_n)\big\}_{n\geq 1}\Big) \leq e^{-\rho T}\chi_{s,V}\Big(\big\{(\overline{u}_n,\overline{v}_n)\big\}_{n\geq 1}\Big)$$

and the set $\big\{(\overline{u}_n,\overline{v}_n)\big\}_{n\geq 1}$ is relatively compact. Hence, we may extract a subsequence $\big((\overline{u}_{n_k},\overline{v}_{n_k})\big)_{k\geq 1}$ such that $(\overline{u}_{n_k},\overline{v}_{n_k}) \to (\overline{u}_0,\overline{v}_0)$ as $k\to\infty$, for some $(\overline{u}_0,\overline{v}_0)\in\mathbb{X}$.

By the continuous dependence of solutions on the initial conditions – see Proposition 1.2.1 (i), we get $(u_{n_k}(t),v_{n_k}(t)) \to (u_0(t),v_0(t))$ as $k\to\infty$, uniformly on bounded subsets of $[0,+\infty)$, where $(u_0,v_0):[0,+\infty)\to\mathbb{X}$ is the mild solution of

$$\begin{cases}(\dot{u}(t),\dot{v}(t)) = \mathbb{A}(u(t),v(t)), & t>0,\\ (u(0),v(0)) = (\overline{u}_0,\overline{v}_0).\end{cases}$$

Consequently, (u_0,v_0) is T-periodic, thus, by virtue of Lemma 4.2.3, $u_0(t)=\varphi$ and $v_0(t)=0$, for $t\geq 0$, with $\varphi\in\operatorname{Ker}(-\Delta+\boldsymbol{V})$. This completes the proof of the lemma. □

Lemma 4.4.3. *Assume that $G:[0,+\infty)\times H^1(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ is a continuous mapping and $M>0$ is such that $\|G(t,u)\|_{L^2}\leq M$, for all $t\geq 0$ and $u\in H^1(\mathbb{R}^N)$. Then there exists $R>0$ such that, for any $(\overline{u},\overline{v})\in\mathbb{X}$ and a T-periodic mild solution $(u,v):[0,+\infty)\to\mathbb{X}$ of*

$$\begin{cases}(\dot{u}(t),\dot{v}(t)) = \mathbb{A}(u(t),v(t)) + \big(0,G(t,u(t))\big), & t>0,\\ (u(0),v(0)) = (\overline{u},\overline{v}),\end{cases} \tag{4.55}$$

the inequality is satisfied

$$\|\widetilde{\mathbb{P}}(u(t),v(t))\|_{\mathbb{X}} \leq R \quad \text{for } t\in[0,T].$$

Proof. Let $(\overline{u},\overline{v})\in\mathbb{X}$ and $(u,v):[0,+\infty)\to\mathbb{X}$ be a T-periodic mild solution of (4.55). Put

$$C_1 = \|\widetilde{\mathbb{P}}\|_{\mathcal{L}(\mathbb{X})} \quad\text{and}\quad C_2 = \sup_{t\in[0,T]}\|e^{t\mathbb{A}}\|_{\mathcal{L}(\mathbb{X})}.$$

Recall that, by Remark 2.4.2, $\widetilde{\mathbb{P}}$ commutes with $e^{t\mathbb{A}}$, thus, in view of the Duhamel formula (see (3.45) in Section 3.4), we obtain, for $t\in[0,T]$,

$$\widetilde{\mathbb{P}}(u(t),v(t)) = e^{t\mathbb{A}}\widetilde{\mathbb{P}}(\overline{u},\overline{v}) + \widetilde{\mathbb{P}}\int_0^t e^{(t-\tau)\mathbb{A}}\big(0,G(\tau,u(\tau))\big)\,d\tau.$$

Observe that, for all $t\in[0,T]$,

$$\begin{aligned}\left\|\widetilde{\mathbb{P}}\int_0^t e^{(t-\tau)\mathbb{A}}\big(0,G(\tau,u(\tau))\big)\,d\tau\right\|_{\mathbb{X}} &\leq C_1\int_0^t C_2\,\big\|\big(0,G(\tau,u(\tau))\big)\big\|_{\mathbb{X}}\,d\tau\\ &\leq C_1\int_0^t C_2\,M\,d\tau \leq C_1\,T\,C_2\,M =: C.\end{aligned}$$

From the T-periodicity of (u,v) we deduce that

$$(I-e^{T\mathbb{A}})\widetilde{\mathbb{P}}(\overline{u},\overline{v}) = \widetilde{\mathbb{P}}\int_0^T e^{(T-\tau)\mathbb{A}}\big(0,G(\tau,u(\tau))\big)\,d\tau.$$

In the light of Lemma 2.4.1 (iv), $1 \in \rho\left(e^{T\mathbb{A}}\big|_{\widetilde{\mathbb{X}}}\right)$. Therefore, $(I - e^{T\mathbb{A}})^{-1} \in \mathcal{L}(\widetilde{\mathbb{X}})$ and let

$$C_3 = \|(I - e^{T\mathbb{A}})^{-1}\|_{\mathcal{L}(\widetilde{\mathbb{X}})}.$$

Hence, we arrive at

$$\begin{aligned}\|\widetilde{\mathbb{P}}(\overline{u}, \overline{v})\|_{\mathbb{X}} &= \left\|(I - e^{T\mathbb{A}})^{-1}\, \widetilde{\mathbb{P}} \int_0^T e^{(T-\tau)\mathbb{A}}\big(0, G(\tau, u(\tau))\big)\, d\tau\right\|_{\mathbb{X}} \\ &\le C_3 \left\|\widetilde{\mathbb{P}} \int_0^T e^{(T-\tau)\mathbb{A}}\big(0, G(\tau, u(\tau))\big)\, d\tau\right\|_{\mathbb{X}} \le C_3\, C.\end{aligned}$$

Finally, we obtain, for any $t \in [0, T]$,

$$\begin{aligned}\|\widetilde{\mathbb{P}}(u(t), v(t))\|_{\mathbb{X}} &\le \|e^{t\mathbb{A}}\widetilde{\mathbb{P}}(\overline{u}, \overline{v})\|_{\mathbb{X}} + \left\|\widetilde{\mathbb{P}} \int_0^t e^{(t-\tau)\mathbb{A}}\big(0, G(\tau, u(\tau))\big)\, d\tau\right\|_{\mathbb{X}} \\ &\le C_2\, \|\widetilde{\mathbb{P}}(\overline{u}, \overline{v})\|_{\mathbb{X}} + C \le C_2\, C_3\, C + C\end{aligned}$$

and the proof is completed. □

Proof of Theorem 4.4.1. We claim that the assumption (b) of Theorem 4.2.4 is satisfied, that is, there exists $R_\infty > 0$ such that,

$$\text{for any } \varepsilon \in (0, 1) \text{ and a } T\text{-periodic mild solution of (4.23), } \|(u(0), v(0))\|_{\mathbb{X}} < R_\infty. \tag{4.56}$$

Indeed, suppose to the contrary, i.e., that there are sequences $(\varepsilon_n)_{n\ge1}$ in $(0, 1)$ and $((\overline{u}_n, \overline{v}_n))_{n\ge1}$ in $\mathbb{X}$ such that $\|(\overline{u}_n, \overline{v}_n)\|_{\mathbb{X}} \to \infty$, and, for any $n \ge 1$, there exists a T-periodic mild solution $(u_n, v_n) : [0, +\infty) \to \mathbb{X}$ of (4.23) with $\varepsilon = \varepsilon_n$ and $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$.

For any $n \ge 1$, let us define $(p_n, q_n) : [0, +\infty) \to \mathbb{X}_0$, $(p_n(t), q_n(t)) = \mathbb{P}_0(u_n(t), v_n(t))$, for $t \ge 0$. Obviously, (p_n, q_n), $n \ge 1$, are T-periodic. Furthermore, $\mathbb{P}_0$ commutes with $e^{t\mathbb{A}}$ (cf. Proposition 2.3.6 (iii)), thus, based on the Duhamel formula and the property that $\mathbb{X}_0$ is finite-dimensional, we infer that (p_n, q_n), $n \ge 1$, are differentiable on $[0, +\infty)$ and solve the ordinary differential system

$$\begin{cases}\dot{p}_n(t) = q_n(t), & t \ge 0\\ \dot{q}_n(t) = -\beta q_n(t) + \varepsilon_n P_0 F(t, u_n(t)), & t \ge 0.\end{cases} \tag{4.57}$$

This yields, for any $t \ge 0$ and $n \ge 1$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\beta p_n(t) + q_n(t)\|_{L^2}^2 &= \langle \beta \dot{p}_n(t) + \dot{q}_n(t), \beta p_n(t) + q_n(t)\rangle_{L^2} \\ &= \varepsilon_n \langle P_0 F(t, u_n(t)), \beta p_n(t) + q_n(t)\rangle_{L^2} \\ &= \varepsilon_n \langle F(t, u_n(t)), P_0\big(\beta p_n(t) + q_n(t)\big)\rangle_{L^2} \\ &= \varepsilon_n \beta \langle F(t, u_n(t)), p_n(t) + \beta^{-1} q_n(t)\rangle_{L^2}.\end{aligned}$$

Hence, by T-periodicity,

$$\int_0^T \langle F(t, u_n(t)), p_n(t) + \beta^{-1} q_n(t)\rangle_{L^2}\, dt = 0 \quad \text{for } n \ge 1. \tag{4.58}$$

From (4.57) we also obtain, for $t \ge 0$ and $n \ge 1$,

$$\frac{d}{dt}\big(e^{\beta t} q_n(t)\big) = \varepsilon_n e^{\beta t} P_0 F(t, u_n(t)).$$

This yields, by T-periodicity, for $t \ge 0$ and $n \ge 1$,

$$\begin{aligned}\varepsilon_n \int_t^{t+T} e^{\beta s} P_0 F(s, u_n(s))\, ds &= e^{\beta(t+T)} q_n(t+T) - e^{\beta t} q_n(t) \\ &= q_n(t)(e^{\beta(t+T)} - e^{\beta t}).\end{aligned}$$

Since f is bounded by $m \in L^2(\mathbb{R}^N)$, we get, for all $t \geq 0$ and $n \geq 1$,

$$\begin{aligned} \|q_n(t)\|_{L^2} &\leq \varepsilon_n (e^{\beta(t+T)} - e^{\beta t})^{-1} \int_t^{t+T} e^{\beta s} \|P_0 F(s, u_n(s))\|_{L^2}\, ds \\ &\leq \varepsilon_n (e^{\beta(t+T)} - e^{\beta t})^{-1} \int_t^{t+T} e^{\beta s} \|F(s, u_n(s))\|_{L^2}\, ds \\ &\leq \varepsilon_n (e^{\beta(t+T)} - e^{\beta t})^{-1} \int_t^{t+T} e^{\beta s} \|m\|_{L^2}\, ds = \varepsilon_n \|m\|_{L^2}/\beta \leq \|m\|_{L^2}/\beta. \end{aligned} \tag{4.59}$$

Now, let $\mu_n = \|(\overline{u}_n, \overline{v}_n)\|_{\mathbb{X}}$ and $(\widetilde{u}_n, \widetilde{v}_n) : [0, +\infty) \to \mathbb{X}$, $n \geq 1$, be given by $(\widetilde{u}_n(t), \widetilde{v}_n(t)) = \mu_n^{-1}(u_n(t), v_n(t))$, for $t \geq 0$. Then we easily see that $(\widetilde{u}_n, \widetilde{v}_n)$, $n \geq 1$, are T-periodic mild solutions of

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \varepsilon_n \mu_n^{-1} \big(0, F(t, \mu_n u(t))\big), \quad t > 0, \\ (u(0), v(0)) = \mu_n^{-1}(\overline{u}_n, \overline{v}_n). \end{cases}$$

Notice that $\|(\widetilde{u}_n(0), \widetilde{v}_n(0))\|_{\mathbb{X}} = 1$, for all $n \geq 1$, and, given that f is bounded by $m \in L^2(\mathbb{R}^N)$,

$$\varepsilon_n \mu_n^{-1} \big(0, F(t, \mu_n \widetilde{u}_n(t))\big) \to 0 \ \text{ in } C([0,T], \mathbb{X}) \text{ as } n \to \infty.$$

Hence, due to Lemma 4.4.2, we may assume that

$$(\widetilde{u}_n, \widetilde{v}_n) \to (\varphi, 0) \quad \text{in } C([0,T], \mathbb{X}), \text{ as } n \to \infty$$

with some $\varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V})$ such that $\|\varphi\|_{H^1} = 1$.

Let us assume that one of the conditions $(LL)_\pm$ is satisfied. Based on Lemma 4.3.9, we have either

$$\liminf_{n\to\infty} \int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), P_0 \widetilde{u}_n(t) \rangle_{L^2}\, dt \geq \int_0^T \int_{\{\varphi > 0\}} \check{f}_+ \varphi \, dx\, dt + \int_0^T \int_{\{\varphi < 0\}} \hat{f}_- \varphi\, dx\, dt > 0 \tag{4.60}$$

or

$$\limsup_{n\to\infty} \int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), P_0 \widetilde{u}_n(t) \rangle_{L^2}\, dt \leq \int_0^T \int_{\{\varphi > 0\}} \hat{f}_+ \varphi \, dx\, dt + \int_0^T \int_{\{\varphi < 0\}} \check{f}_- \varphi\, dx\, dt < 0. \tag{4.61}$$

On the other hand, dividing (4.58) by μ_n, and using (4.59), we get

$$\begin{aligned} \int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), P_0 \widetilde{u}_n(t) \rangle_{L^2}\, dt &= \mu_n^{-1} \int_0^T \langle F(t, u_n(t)), p_n(t) \rangle_{L^2}\, dt \\ &= -\mu_n^{-1} \int_0^T \langle F(t, u_n(t)), \beta^{-1} q_n(t) \rangle_{L^2}\, dt \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which contradicts either (4.60) or (4.61). This shows the existence of $R_\infty > 0$ such that (4.56) is satisfied. Consequently, increasing R_∞ if necessary, Lemma 4.3.12 implies that the assumption (a) of Theorem 4.2.4 is fulfilled. Hence, we may apply Theorem 4.2.4 to get the existence of a T-periodic mild solution of (4.3).

Now, we assume that $N \geq 3$ and either $(SR)_+$ or $(SR)_-$ holds. In view of Lemma 4.3.10, we have either

$$\liminf_{n\to\infty} \int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), \mu_n \widetilde{u}_n(t) \rangle_{L^2}\, dt \geq \int_0^T \int_{\{\varphi > 0\}} k_+ \, dx\, dt + \int_0^T \int_{\{\varphi < 0\}} k_-\, dx\, dt > 0 \tag{4.62}$$

or

$$\limsup_{n\to\infty} \int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), \mu_n \widetilde{u}_n(t) \rangle_{L^2}\, dt \leq \int_0^T \int_{\{\varphi > 0\}} k_+ \, dx\, dt + \int_0^T \int_{\{\varphi < 0\}} k_-\, dx\, dt < 0. \tag{4.63}$$

On the other hand, put $w_n : [0, +\infty) \to H^1(\mathbb{R}^N)$, $w_n(t) = u_n(t) - p_n(t)$, $t \in [0, +\infty)$, $n \geq 1$. Then

$$\widetilde{\mathbb{P}}(u_n(t), v_n(t)) = (\widetilde{P}u_n(t), \widetilde{P}v_n(t)) = (w_n(t), \widetilde{P}v_n(t)) \text{ for any } t \geq 0 \text{ and } n \geq 1,$$

because $\widetilde{\mathbb{P}}(u, v) = (\widetilde{P}u, \widetilde{P}v)$, for all $(u, v) \in \mathbb{X}$. Thus, in the light of Lemma 4.4.3, there is $R > 0$ such that

$$\|w_n(t)\|_{H^1} \leq \|\widetilde{\mathbb{P}}(u_n(t), v_n(t))\|_{\mathbb{X}} \leq R, \quad \text{for all } t \in [0, T] \quad \text{and} \quad n \geq 1.$$

Since $q_n([0, T]) \subset X_0$ and X_0 is a finite-dimensional subspace, the $L^2(\mathbb{R}^N)$-norm and $H^1(\mathbb{R}^N)$-norm are equivalent on X_0. Thus, from (4.59) we deduce that

$$(w_n - \beta^{-1} q_n)_{n \geq 1} \text{ is bounded in } C([0, T], H^1(\mathbb{R}^N)).$$

Hence, combining Remark 4.3.3 with Lemma 4.3.11, we obtain

$$\int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), w_n(t) - \beta^{-1} q_n(t)\rangle_{L^2} \, dt \to 0 \quad \text{as } n \to \infty.$$

Consequently, it follows from (4.58) that

$$\begin{aligned}
\int_0^T \langle F(t, \mu_n \widetilde{u}_n(t)), \mu_n \widetilde{u}_n(t)\rangle_{L^2} \, dt &= \int_0^T \langle F(t, u_n(t)), u_n(t)\rangle_{L^2} \, dt \\
&= \int_0^T \langle F(t, u_n(t)), p_n(t) + w_n(t)\rangle_{L^2} \, dt \\
&= \int_0^T \langle F(t, u_n(t)), w_n(t) - \beta^{-1} q_n(t)\rangle_{L^2} \, dt \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which contradicts either (4.62) or (4.63). This shows the existence of $R_\infty > 0$ such that (4.56) is satisfied. Consequently, increasing R_∞ if necessary, Lemma 4.3.12 implies again that the assumption (a) of Theorem 4.2.4 is fulfilled. Therefore, we may apply Theorem 4.2.4 once more to get the existence of a T-periodic mild solution of (4.3). Hence, the proof is completed. □

4.5 Applications to PDEs

We study the following equation

$$u_{tt} + \beta u_t = \Delta u + \frac{a}{|x|} u + \lambda_k u + g(t, x) b\big(v(t, x)(u - r(x))\big), \quad t > 0, \quad x \in \mathbb{R}^3 \tag{4.64}$$

where $\beta > 0$ is a damping coefficient, $a > 0$ is a constant, and λ_k is the k-th eigenvalue of the Schrödinger operator

$$-\Delta - \frac{a}{|x|} \tag{4.65}$$

– see Example 2.1.15, $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $v : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$, and $r : \mathbb{R}^3 \to \mathbb{R}$ are certain functions.

We present results on the existence of periodic solutions of (4.64) in two theorems. The first theorem concerns the Landesman-Lazer type conditions.

Theorem 4.5.1. *Assume that g is a T-periodic Carathéodory function such that there exists $g_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ satisfying*

$$|g(t, x)| \leq g_0(x) \quad \textit{for almost every } x \in \mathbb{R}^3 \textit{ and all } t \geq 0, \tag{4.66}$$

b is bounded and satisfies the Lipschitz condition, v is a T-periodic Carathéodory function, which is bounded and positive, i.e., $v(t, x) > 0$ for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$, and r is measurable.

(i) If g fulfills the condition

$$(G)_+ \quad \begin{cases} g(t,x) \geq 0 & \textit{for almost every } x \in \mathbb{R}^3 \textit{ and all } t \geq 0, \\ g(t,x) > 0 & \textit{for all } x \in E \textit{ and } t \geq 0, \textit{ for some } E \subset \mathbb{R}^3 \textit{ of positive measure,} \end{cases}$$

and

$$b_\pm = \lim_{u \to \pm\infty} b(u) \tag{4.67}$$

where $b_\pm$ are real numbers such that $b_+ b_- < 0$, then equation (4.64) *admits a T-periodic mild solution.*

(ii) If g fulfills the condition

$$(G)_- \quad \begin{cases} g(t,x) \leq 0 & \textit{for almost every } x \in \mathbb{R}^3 \textit{ and all } t \geq 0, \\ g(t,x) < 0 & \textit{for all } x \in E \textit{ and } t \geq 0, \textit{ for some } E \subset \mathbb{R}^3 \textit{ of positive measure} \end{cases}$$

and $b_\pm$ given by (4.67) *are as in the point (i), then equation* (4.64) *admits a T-periodic mild solution.*

Remark 4.5.2. As an example of a function $b : \mathbb{R} \to \mathbb{R}$ satisfying assumptions of Theorem 4.5.1 we can take

$$b(u) = \arctan(u) \quad \text{or} \quad b(u) = -\arctan(u).$$

Indeed, if $b(u) = \arctan(u)$, then $b_+ = \lim_{u\to+\infty} b(u) = \pi/2$ and $b_- = \lim_{u\to-\infty} b(u) = -\pi/2$, and, if $b(u) = -\arctan(u)$, then $b_+ = -\pi/2$ and $b_- = \pi/2$. Additionally, if r is not an eigenfunction of the Schrödinger operator $-\Delta - a/|x|$ with the eigenvalue λ_k, then equation (4.64) does not admit a constant T-periodic mild solution given by $(u(t), v(t)) = (r, 0)$. □

Proof of Theorem 4.5.1. Let us define $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by the formula

$$V(x) = -\frac{a}{|x|} - \lambda_k \tag{4.68}$$

and $f : [0, +\infty) \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by the formula

$$f(t,x,u) = g(t,x) b\big(v(t,x)(u - r(x))\big). \tag{4.69}$$

In view of Example 2.1.1, V is a Kato-Rellich type potential such that $\varrho(V_\infty) = -\lambda_k > 0$. We see that f is a T-periodic Carathéodory function. We now verify that f satisfies condition (f1). Indeed, for almost every $x \in \mathbb{R}^3$, for all $t \geq 0$, and for all u_1, $u_2 \in \mathbb{R}$, one has

$$\begin{aligned} |f(t,x,u_1) - f(t,x,u_2)| &\leq |g(t,x)| \, \big| b\big(v(t,x)(u_1 - r(x))\big) - b\big(v(t,x)(u_2 - r(x))\big) \big| \\ &\leq g_0(x) \, L_b \, M_v \, |u_1 - u_2| \end{aligned}$$

where g_0 is as in (4.66), L_b is the Lipschitz constant of b, and M_v is a bound for v. Since $g_0 \in L^3(\mathbb{R}^3)$, it is a Kato-Rellich type function. Hence f satisfies (f1) with $l = L_b M_v g_0$. As b is bounded and g satisfies (4.66), it follows that f satisfies (f2)$'$. By Example 2.1.15, one has

$$\sigma_{\text{disc}}(-\Delta + \boldsymbol{V}) = \{\lambda_n - \lambda_k : n \geq 1\} \quad \text{and} \quad \sigma_{\text{ess}}(-\Delta + \boldsymbol{V}) = [-\lambda_k, +\infty)$$

where λ_n, $n \geq 1$, are the eigenvalues of the Schrödinger operator (4.65). In particular, we have $0 \in \sigma_{\text{disc}}(-\Delta + \boldsymbol{V})$, and thus $\operatorname{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\}$.

We turn to the proof of the point (i). From $(G)_+$ we deduce that function $G : \mathbb{R}^3 \to \mathbb{R}$,

$$G(x) = \int_0^T g(t,x) \, dt \tag{4.70}$$

is non-negative, i.e., $G(x) \geq 0$ for almost every $x \in \mathbb{R}^3$, and $G|_E > 0$ for some set $E \subset \mathbb{R}^3$ of positive measure. Since $b_\pm$ given by (4.67) are real numbers, one has

$$\widehat{h}_\pm(x) = \int_0^T \limsup_{u\to\pm\infty} f(t,x,u)\,dt = \int_0^T \limsup_{u\to\pm\infty} g(t,x)\; b\big(v(t,x)(u-r(x))\big)\,dt = b_\pm G(x) \quad (4.71)$$

and similarly

$$\check{h}_\pm(x) = b_\pm G(x). \quad (4.72)$$

Since $b_+b_- < 0$, we have either $b_+ > 0$ and $b_- < 0$, or $b_+ < 0$ and $b_- > 0$. If the former holds, then, by (4.71) and (4.72), we obtain

$$\check{h}_+(x) \geq 0 \ \text{ for almost every } x \in \mathbb{R}^3, \quad \text{and} \quad \check{h}_+\Big|_E > 0$$

and

$$\widehat{h}_-(x) \leq 0 \ \text{ for almost every } x \in \mathbb{R}^3, \quad \text{and} \quad \widehat{h}_-\Big|_E < 0.$$

Therefore, the Landesman-Lazer type condition $(LL)'_+$ is satisfied and, by Theorem 4.4.1 (i), there exists a T-periodic mild solution of (4.64).

Next, if the later case occurs, i.e., $b_+ < 0$ and $b_- > 0$, we see that, again by (4.71) and (4.72),

$$\widehat{h}_+(x) \leq 0 \ \text{ for almost every } x \in \mathbb{R}^3, \quad \text{and} \quad \widehat{h}_+\Big|_E < 0$$

and

$$\check{h}_-(x) \geq 0 \ \text{ for almost every } x \in \mathbb{R}^3, \quad \text{and} \quad \check{h}_-\Big|_E > 0.$$

This means that the Landesman-Lazer type condition $(LL)'_-$ is satisfied. This allows us to apply Theorem 4.4.1 (i) obtaining the existence of a T-periodic mild solution of (4.64).

Now, we prove the assertion (ii). By the assumed condition $(G)_-$, function $G : \mathbb{R}^3 \to \mathbb{R}$, given by (4.70), satisfies $G(x) \leq 0$, for almost every $x \in \mathbb{R}^3$, and $G|_E < 0$ for some set $E \subset \mathbb{R}^3$ of positive measure. By the assumption, $b_+b_- < 0$, hence, either $b_+ > 0$ and $b_- < 0$, or $b_+ < 0$ and $b_- > 0$. In view of the formulas (4.71) and (4.72), if the former possibility holds, then the Landesman-Lazer type condition $(LL)'_-$ is satisfied, and if the later possibility holds, then the Landesman-Lazer type condition $(LL)'_+$ is satisfied. Then, by Theorem 4.4.1 (i), there exists a T-periodic mild solution of (4.64). □

We present now the second theorem concerning the strong resonance type conditions.

Theorem 4.5.3. *Assume that functions g, b, and v are as in Theorem 4.5.1 and $r \in L^2(\mathbb{R}^3)$ is such that $r(x) \geq 0$, for almost every $x \in \mathbb{R}^3$.*

(i) Suppose that g satisfies condition $(G)_+$ and

$$c_\pm = \lim_{u\to\pm\infty} ub(u) \quad (4.73)$$

are real numbers. If either

$$c_+,\ c_- > 0 \quad \text{and} \quad ub(u) \geq 0 \text{ for all } u \in \mathbb{R},$$

or

$$c_+,\ c_- < 0 \quad \text{and} \quad ub(u) \leq 0 \text{ for all } u \in \mathbb{R},$$

then equation (4.64) admits a T-periodic mild solution.

(ii) If g satisfies condition $(G)_-$ and numbers $c_\pm$ given by (4.73) are as in the point (i), then equation (4.64) admits a T-periodic mild solution.

Remark 4.5.4. As an example of a function $b : \mathbb{R} \to \mathbb{R}$ satisfying assumptions of Theorem 4.5.3 we can take

$$b(u) = \frac{u}{1+u^2} \quad \text{or} \quad b(u) = -\frac{u}{1+u^2}.$$

Indeed, if $b(u) = u/(1+u^2)$, then $c_+ = c_- = 1$ and $ub(u) \geq 0$ for all $u \in \mathbb{R}$. On the other hand, if $b(u) = -u/(1+u^2)$, then $c_+ = c_- = -1$ and $ub(u) \leq 0$ for all $u \in \mathbb{R}$. Additionally, if r is not an eigenfunction of the Schrödinger operator $-\Delta - a/|x|$ with the eigenvalue λ_k, then equation (4.64) does not admit a constant T-periodic mild solution given by $(u(t), v(t)) = (r, 0)$. □

Proof of Theorem 4.5.3. It follows from the proof of Theorem 4.5.1 that the function V given by (4.68) is a Kato-Rellich type potential and the function f given by (4.69) is a T-periodic Carathéodory function satisfying (f1) and (f2)$'$. In addition, $\mathrm{Ker}(-\Delta + \boldsymbol{V}) \neq \{0\}$.

We prove the assertion (i). Since, by the assumption, $(G)_+$ holds and v is positive, we obtain

$$\frac{g(t,x)}{v(t,x)} > 0 \quad \text{for all } x \in E \text{ and all } t \geq 0, \tag{4.74}$$

for a set of positive measure $E \subset \mathbb{R}^3$. Note that, for almost every $x \in \mathbb{R}^3$, all $t \geq 0$, and all $u \in \mathbb{R}$,

$$\begin{aligned} uf(t,x,u) &= u\, g(t,x)\, b\big(v(t,x)(u - r(x))\big) \\ &= \frac{g(t,x)}{v(t,x)}\, v(t,x)\, (u - r(x))\, b\big(v(t,x)(u - r(x))\big) \\ &\quad + g(t,x)\, r(x)\, b\big(v(t,x)(u - r(x))\big). \end{aligned} \tag{4.75}$$

Moreover, since, by the assumption, $c_\pm$ are real numbers, we have $\lim_{u\to\pm\infty} b(u) = 0$. Hence, for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$,

$$k_\pm(t,x) = \lim_{u\to\pm\infty} uf(t,x,u) = \frac{c_\pm\, g(t,x)}{v(t,x)} \tag{4.76}$$

are real numbers.

Suppose that $c_+ > 0$, $c_- > 0$ and $ub(u) \geq 0$ for $u \in \mathbb{R}$. We have then that

$$b(u) \geq 0 \quad \text{for all } u \geq 0 \quad \text{and} \quad b(u) \leq 0 \quad \text{for all } u \leq 0.$$

Let $x \in \mathbb{R}^3$. If $u \geq r(x)$, then, for all $t \geq 0$,

$$v(t,x)(u - r(x)) \geq 0,$$

hence

$$b\big(v(t,x)(u - r(x))\big) \geq 0. \tag{4.77}$$

Similarly, if $u \leq -r(x)$, then, for all $t \geq 0$,

$$b\big(v(t,x)(u - r(x))\big) \leq 0. \tag{4.78}$$

Therefore, we have

$$uf(t,x,u) \geq 0 \quad \text{for all } |u| \geq r(x) \text{ and all } t \geq 0. \tag{4.79}$$

By (4.74) and (4.76),

$$\int_0^T k_\pm(t,x)\, dt > 0 \quad \text{for all } x \in E. \tag{4.80}$$

This means that the strong resonance type condition $(SR)'_+$ holds and, by Theorem 4.4.1 (ii), there exists a T-periodic mild solution of (4.64).

Now, assume that $c_+ < 0$, $c_- < 0$ and $ub(u) \leq 0$ for $u \in \mathbb{R}$. Hence,

$$b(u) \leq 0 \quad \text{for all } u \geq 0 \quad \text{and} \quad b(u) \geq 0 \quad \text{for all } u \leq 0.$$

Similarly as before, we show that

$$uf(t,x,u) \leq 0 \quad \text{for all } |u| \geq r(x) \text{ and all } t \geq 0. \tag{4.81}$$

Moreover, from (4.74) and (4.76) we deduce that

$$\int_0^T k_\pm(t,x)\,dt < 0 \quad \text{for all } x \in E. \tag{4.82}$$

Therefore, the strong resonance type condition $(SR)'_-$ is satisfied and, using again Theorem 4.4.1 (ii), we obtain that equation (4.64) admits a T-periodic mild solution.

We prove now the assertion (ii). From the assumed condition $(G)_-$ and the assumption that v is positive we deduce that

$$\frac{g(t,x)}{v(t,x)} < 0 \quad \text{for all } x \in E \text{ and all } t \geq 0. \tag{4.83}$$

Suppose that $ub(u) \geq 0$, for all $u \in \mathbb{R}$, and $c_+,\ c_- > 0$. Let $x \in \mathbb{R}^3$ and $t \geq 0$. From (4.77) and (4.78) it follows that, for almost every $x \in \mathbb{R}^3$,

$$ub\big(v(t,x)(u - r(x))\big) \geq 0 \quad \text{for all } |u| \geq r(x) \text{ and all } t \geq 0.$$

By the assumption, $g(t,x) \leq 0$, for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$, which implies (4.81). Moreover, (4.76) and (4.83) yield (4.82). Thus, the strong resonance type condition $(SR)'_-$ is satisfied and we can use Theorem 4.4.1 (ii) to get the existence of a T-periodic mild solution of (4.64).

Now, let $ub(u) \leq 0$, for all $u \in \mathbb{R}$, and $c_+,\ c_- < 0$. Similarly as before, we obtain that, for almost every $x \in \mathbb{R}^3$,

$$ub\big(v(t,x)(u - r(x))\big) \leq 0 \quad \text{for all } |u| \geq r(x) \text{ and all } t \geq 0.$$

Since $g(t,x) \leq 0$, for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$, we arrive at (4.79). From (4.76) and (4.83) we deduce (4.80). This means that the strong resonance type condition $(SR)'_+$ holds and, based again on Theorem 4.4.1 (ii), equation (4.64) admits a T-periodic mild solution. The proof is completed. □

Chapter 5

Periodic solutions for non-resonant equations

In this chapter, we study periodic solutions of the nonlinear damped wave equation (see (1)):

$$u_{tt} + \beta u_t = \Delta u - V(x)u + f(t, x, u), \quad t > 0, \ x \in \mathbb{R}^N \tag{5.1}$$

under non-resonance conditions – that is, when the kernels of $-\Delta + \boldsymbol{V}$, perturbed by the linearizations of f at infinity and at zero, are trivial. Here $\beta > 0$ is a damping coefficient, $V : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type potential (see (5a) and (5b)) whose $L^\infty(\mathbb{R}^N)$-part has a positive asymptotic bottom (see (7)). The nonlinear term $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed to be a T-periodic Carathéodory function satisfying conditions (f1) and (f2) (cf. condition (P) on p. ix, condition (C) on p. viii, condition (f1) on p. ix, and condition (f2) on p. ix).

As in the resonant case, we seek T-periodic mild solutions of (5.1), i.e., T-periodic mild solutions of the associated evolution equation:

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F(t, u(t))\big), \quad t > 0 \tag{5.2}$$

where $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}$ is the damped wave operator (see Section 2.2), and $F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the function f (see Section 3.1).

Section 5.1 develops averaging principles for nonlinear damped wave equations. First, we prove an averaging principle for arbitrary sequences of equations. Next, we consider a sequence of frequency changing problems with t_n-periodic solutions, where $t_n \to 0^+$, and we prove that the sequence of periodic points contains a converging subsequence and a corresponding sequence of solutions converges uniformly to a stationary solution.

In Section 5.2, we introduce the index averaging formula, which states that, for sufficiently small times, the index of the translation along trajectories operator associated with the frequency changing nonlinear damped wave equation coincides with the index of the translation operator associated with the autonomous equation with the time-averaged nonlinearity. From this formula, we derive the continuation principle.

Section 5.3 establishes *a priori estimates*: we prove that the absence of nonzero periodic solutions of a linearized equation implies certain bounds for the starting points of the frequency changing nonlinear damped wave equation. This allows us to apply the continuation principle.

Section 5.4 presents index formulae for the translation along trajectories operator associated with the autonomous nonlinear damped wave equation under non-resonance conditions.

Section 5.5 contains the proof of Theorem 5.5.1 on the existence of T-periodic solutions to the equation (5.1) under non-resonance conditions. The chapter concludes with Section 5.6, which applies this theorem to the equation (5.1) with the Coulomb potential on $\mathbb{R}^3$ and specific classes of nonlinearities.

5.1 Averaging principles for nonlinear damped wave equations

We consider the evolution equation (5.2) with a parameter $\varepsilon \in (0,1]$:

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, F(t/\varepsilon, u(t))\big), \quad t > 0 \tag{5.3}$$

and the autonomous evolution equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, \widehat{F}(u(t))\big), \quad t > 0 \tag{5.4}$$

where $\widehat{F} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the function $\widehat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{f}(x,u) = \frac{1}{T}\int_0^T f(t,x,u)\, dt \quad \text{for almost every } x \in \mathbb{R}^N, \text{ and all } u \in \mathbb{R}. \tag{5.5}$$

By Proposition 3.1.2 (ii), function $\widehat{f}$ satisfies the Carathéodory condition, (f1) with the same function l as f, and (f2) with the same function m_0 as f, and hence the mapping $\widehat{F}$ is well-defined and satisfies (**F1**) (see p. 48).

Proposition 5.1.1. *Let $\big((\overline{u}_n, \overline{v}_n)\big)_{n\ge 1}$ be a sequence in $\mathbb{X}$ such that $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$ as $n \to \infty$, and $(\varepsilon_n)_{n\ge1}$ be a sequence in $(0,+\infty)$ such that $\varepsilon_n \xrightarrow{n\to\infty} \varepsilon_0$ in $[0,+\infty)$. Assume that $f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \ge 0$, be T-periodic Carathéodory functions satisfying conditions (f1) and (f2) with common functions l and m_0, and be such that $f_n \to f_0$ in the sense of condition (f0) (see p. 50).*

Let $(u_n, v_n) : [0,+\infty) \to \mathbb{X}$, $n \ge 1$, are the mild solutions of the equation (5.3) with $F = F_n$, $n \ge 1$, being the Nemytskii operators associated with the functions f_n, $n \ge 1$, $\varepsilon = \varepsilon_n$, and the initial conditions $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$.

(i) Suppose that $\varepsilon_0 > 0$ and let $(u_0, v_0) : [0,+\infty) \to \mathbb{X}$ be the mild solution of the problem (5.3) with $F = F_0$ being the Nemytskii operator associated with the function f_0, $\varepsilon = \varepsilon_0$, and the initial condition $(u_0(0), v_0(0)) = (\overline{u}_0, \overline{v}_0)$. Then, the convergence of

$$(u_n(t), v_n(t)) \to (u_0(t), v_0(t)) \quad \text{as } n \to \infty$$

is uniform on bounded subsets of $[0,+\infty)$ with respect to t.

(ii) Assume that $\varepsilon_0 = 0$ and let $(\widehat{u}_0, \widehat{v}_0) : [0,+\infty) \to \mathbb{X}$ be the mild solution of the problem (5.4) with $F = F_0$ and the initial condition $(\widehat{u}_0(0), \widehat{v}_0(0)) = (\overline{u}_0, \overline{v}_0)$. Then, the limit

$$(u_n(t), v_n(t)) \to (\widehat{u}_0(t), \widehat{v}_0(t)) \quad \text{as } n \to \infty$$

holds uniformly on bounded subsets of $[0,+\infty)$ with respect to t.

Proof. (i) We define the mappings $\mathbb{F}_n : [0,+\infty) \times \mathbb{X} \to \mathbb{X}$, $n \ge 0$, as follows

$$\mathbb{F}_n(t,u,v) = \big(0, F_n(t,u)\big) \quad \text{for } t \ge 0 \text{ and } (u,v) \in \mathbb{X}. \tag{5.6}$$

By Proposition 3.1.3 (iii), operators $\mathbb{F}_n$, $n \ge 0$, are continuous and satisfy conditions (F1) and (F2) with common constants, and the set

$$\{\mathbb{F}_n(t,u,v) : t \in D, n \ge 1\}$$

is relatively compact in $\mathbb{X}$, for any bounded $D \subset [0,+\infty)$ and $(u,v) \in \mathbb{X}$. Using Proposition 3.1.3 (iii) once more, we get

$$\int_0^t \mathbb{F}_n(\tau,u,v)\, d\tau \xrightarrow{n\to\infty} \int_0^t \mathbb{F}_0(\tau,u,v)\, d\tau \quad \text{in } \mathbb{X},$$

for all $t > 0$ and $(u, v) \in \mathbb{X}$. Therefore, all the assumptions of Theorem 1.4.1 are satisfied and its application yields the assertion.

(ii) Let us define the mappings $\widehat{\mathbb{F}}_n : \mathbb{X} \to \mathbb{X}$, $n \geq 0$, by

$$\widehat{\mathbb{F}}_n(u, v) = \frac{1}{T} \int_0^T \mathbb{F}_n(t, u, v)\, dt \quad \text{for any } (u, v) \in \mathbb{X},$$

where the operators $\mathbb{F}_n$, $n \geq 0$, are given by (5.6). As the mappings F_n, $n \geq 1$, are T-periodic on time (cf. condition (**F3**) on p. 66), we see that $\mathbb{F}_n$, $n \geq 0$, also have this property (cf. condition (F3) on p. 13). In view of Proposition 3.1.3 (iii), for any $(u, v) \in \mathbb{X}$,

$$\widehat{\mathbb{F}}_n(u, v) \xrightarrow{n\to\infty} \widehat{\mathbb{F}}_0(u, v) \ \text{ in } \mathbb{X}.$$

Thus, all the assumptions of Theorem 1.4.2 are satisfied and its application ends the proof. □

Proposition 5.1.2. *Let $\big((\overline{u}_n, \overline{v}_n)\big)_{n\geq 1}$ be a bounded sequence, and let the sequence $(\varepsilon_n)_{n\geq 1}$ and the functions f_n, $n \geq 0$, be as in Proposition 5.1.1. Assume additionally that the function l from condition* (f1) *satisfies*

$$\widehat{\varrho}(l_\infty) < \sqrt{d}\, \rho \tag{5.7}$$

where $\widehat{\varrho}(l_\infty) \geq 0$ is given by (14)*, $d > 0$ is as defined in* (3.46)*, and $\rho > 0$ is as in* (3.43)*.*

Moreover, suppose that $(u_n, v_n) : [0, +\infty) \to \mathbb{X}$, $n \geq 1$, are the $\varepsilon_n T$-periodic mild solutions of the equation (5.3) *with $F = F_n$, $\varepsilon = \varepsilon_n$, and the initial conditions $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$.*

(i) Suppose that $\varepsilon_0 > 0$. Then, passing to a subsequence if necessary, $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$ as $n \to \infty$. Let $(u_0, v_0) : [0, +\infty) \to \mathbb{X}$ be the mild solution of the problem (5.3) *with $F = F_0$, $\varepsilon = \varepsilon_0$, and the initial condition $(u_0(0), v_0(0)) = (\overline{u}_0, \overline{v}_0)$. Then, the limit*

$$(u_n(t), v_n(t)) \to (u_0(t), v_0(t)) \quad \text{as } n \to \infty \tag{5.8}$$

holds uniformly on bounded subsets of $[0, +\infty)$ with respect to t, and the solution (u_0, v_0) is $\varepsilon_0 T$-periodic.

(ii) Assume that $\varepsilon_0 = 0$. Then, passing to a subsequence if necessary, $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0) \in D(\mathbb{A})$ as $n \to \infty$, and

$$\mathbb{A}(\overline{u}_0, \overline{v}_0) + (0, \widehat{F}_0(\overline{u}_0)) = 0. \tag{5.9}$$

In addition, the convergence

$$(u_n(t), v_n(t)) \to (\overline{u}_0, \overline{v}_0) \quad \text{as } n \to \infty$$

is uniform on bounded subsets of $[0, +\infty)$ with respect to t.

Proof. (i) Observe that the mappings $F_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \geq 1$, are T-periodic, continuous, and satisfy conditions (**F1**) and (**F2**) with common constants. In addition, by Proposition 3.3.1,

$$\chi_{L^2}\Big(\bigcup_{n=1}^{\infty} F_n([0, T] \times U)\Big) \leq \widehat{\varrho}(l_\infty)\chi_{L^2}(U)$$

for any bounded $U \subset H^1(\mathbb{R}^N)$. Therefore, it follows from Proposition 3.5.1 that $\{(\overline{u}_n, \overline{v}_n)\}_{n\geq 1}$ is relatively compact, hence, we may assume that $(\overline{u}_n, \overline{v}_n) \xrightarrow{n\to\infty} (\overline{u}_0, \overline{v}_0)$ for some $(\overline{u}_0, \overline{v}_0) \in \mathbb{X}$. Applying Proposition 5.1.1 (i) we get that the limit (5.8) holds uniformly on bounded subsets of $[0, +\infty)$ with respect to t. We shall show that the solution (u_0, v_0) is $\varepsilon_0 T$-periodic. To this end, we fix some $t \geq 0$. Since, for any $n \geq 1$, the solution (u_n, v_n) is $\varepsilon_n T$-periodic, one has

$$(u_n(t + \varepsilon_n T), v_n(t + \varepsilon_n T)) = (u_n(t), v_n(t)). \tag{5.10}$$

Moreover, the solution (u_0, v_0) is continuous and $t + \varepsilon_n T \xrightarrow{n\to\infty} t + \varepsilon_0 T$, which yields

$$(u_n(t + \varepsilon_n T), v_n(t + \varepsilon_n T)) \to (u_0(t + \varepsilon_0 T), v_0(t + \varepsilon_0 T)) \ \text{ as } n \to \infty.$$

Therefore, from (5.8), (5.10), and the limit above we deduce that

$$(u_0(t + \varepsilon_0 T), v_0(t + \varepsilon_0 T)) = (u_0(t), v_0(t)).$$

Since $t \geq 0$ was arbitrary, this shows that (u_0, v_0) is $\varepsilon_0 T$-periodic.

(ii) We examined in (i) that all the assumptions of Proposition 3.5.1 are satisfied, hence, passing to a subsequence if necessary, $(\overline{u}_n, \overline{v}_n) \xrightarrow{n\to\infty} (\overline{u}_0, \overline{v}_0)$ for some $(\overline{u}_0, \overline{v}_0) \in \mathbb{X}$. Therefore, by Proposition 5.1.1 (ii), $(u_n(t), v_n(t)) \to (\widehat{u}_0(t), \widehat{v}_0(t))$ as $n \to \infty$, uniformly on bounded subsets of $[0, +\infty)$ with respect to t, where $(\widehat{u}_0, \widehat{v}_0) : [0, +\infty) \to \mathbb{X}$ is the mild solution of the problem (5.4) with $F = F_0$ and the initial condition $(\widehat{u}_0(0), \widehat{v}_0(0)) = (\overline{u}_0, \overline{v}_0)$.

We claim that $(\widehat{u}_0(t), \widehat{v}_0(t)) = (\overline{u}_0, \overline{v}_0)$ for any $t > 0$. Indeed, let $t > 0$ and, for any $n \geq 1$, we take $k_n = [t/\varepsilon_n T]$. Then $k_n \varepsilon_n T \to t$ and, using the uniform convergence, we get

$$(u_n(k_n \varepsilon_n T), v_n(k_n \varepsilon_n T)) \to (\widehat{u}_0(t), \widehat{v}_0(t)) \ \text{ as } n \to \infty.$$

On the other hand, by the periodicity,

$$(u_n(k_n \varepsilon_n T), v_n(k_n \varepsilon_n T)) = (\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0) \ \text{ as } n \to \infty.$$

Therefore, $(\widehat{u}_0, \widehat{v}_0)$ is a constant solution of (5.4) with $F = F_0$ and $(\widehat{u}_0(0), \widehat{v}_0(0)) = (\overline{u}_0, \overline{v}_0)$. Then, Lemma 1.2.3 shows that $(\overline{u}_0, \overline{v}_0) \in D(\mathbb{A})$ and

$$\mathbb{A}(\overline{u}_0, \overline{v}_0) + (0, \widehat{F}_0(\overline{u}_0)) = 0.$$

□

5.2 Index averaging formula

Let $\Phi_t^{(\varepsilon)} : \mathbb{X} \to \mathbb{X}$ be the translation along trajectories operator associated with (5.3). Since the Nemytskii operator $F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous and satisfies (**F1**) and (**F2**), we see, based on Theorem 3.4.1 (points (i) and (ii)), that $\Phi_t^{(\varepsilon)}$ is well-defined and continuous, for any $t > 0$ and $\varepsilon \in (0, 1]$. Moreover, if we assume additionally that $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies (f1) with a function l such that (5.7) holds, then, by Proposition 3.3.1,

$$\chi_{L^2}\big(F(D \times U)\big) \leq \widehat{\varrho}(l_\infty)\chi_{L^2}(U) < \sqrt{d}\,\rho\,\chi_{L^2}(U),$$

for any bounded sets $D \subset [0, +\infty)$ and $U \subset H^1(\mathbb{R}^N)$. Therefore, by Theorem 3.4.1 (iii), $\Phi_t^{(\varepsilon)}$ is a k-set contraction, for any $t > 0$ and $\varepsilon \in (0, 1]$. More precisely,

$$\chi_{s,V}(\Phi_t^{(\varepsilon)}(Z)) \leq e^{-(\rho - \widehat{\varrho}(l_\infty)/\sqrt{d})t}\,\chi_{s,V}(Z),$$

for any $t > 0$ and bounded subset $Z \subset \mathbb{X}$, where $\chi_{s,V}$ is the Hausdorff measure of non-compactness in the space $(\mathbb{X}, \|\cdot\|_{s,V})$ and $\|\cdot\|_{s,V}$ is the norm induced by the scalar product $\langle\cdot,\cdot\rangle_{s,V}$ (cf. (2.60) and (2.58)). Let $\widehat{\Phi}_t : \mathbb{X} \to \mathbb{X}$ be the translation along trajectories operator associated with (5.4). Since the Nemytskii operator $\widehat{F} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is continuous and satisfies (**F1**), we see that $\widehat{\Phi}_t$ is well-defined and continuous, for any $t > 0$. Recall that (see Proposition 3.1.2 (ii)) function $\widehat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills conditions (f1) and (f2) with the same functions l and m_0 as f. Therefore, if f satisfies (f1) with l such that (5.7) holds, then $\widehat{\Phi}_t$ is also a k-set contraction, for any $t > 0$.

Now, we shall prove the index averaging formula.

Theorem 5.2.1. *Assume that a T-periodic Carathéodory function $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies* (f1) *and* (f2)*, and function l from* (f1) *is such that* (5.7) *holds, and $Z \subset \mathbb{X}$ is an open bounded set such that*

$$\mathbb{A}(u,v) + (0, \widehat{F}(u)) \neq 0 \quad \textit{for } (u,v) \in D(\mathbb{A}) \cap \partial Z. \tag{5.11}$$

Then, there exists $\varepsilon_0 \in (0,1]$ such that, for all $\varepsilon \in (0,\varepsilon_0]$,

$$\Phi^{(\varepsilon)}_{\varepsilon T}(\overline{u},\overline{v}) \neq (\overline{u},\overline{v}) \quad \textit{and} \quad \widehat{\Phi}_{\varepsilon T}(\overline{u},\overline{v}) \neq (\overline{u},\overline{v}) \quad \textit{for all } (\overline{u},\overline{v}) \in \partial Z,$$

and

$$\mathrm{Ind}_C(\Phi^{(\varepsilon)}_{\varepsilon T}, Z) = \mathrm{Ind}_C(\widehat{\Phi}_{\varepsilon T}, Z). \tag{5.12}$$

Proof. We define $h : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \times [0,1] \to \mathbb{R}$ as

$$h(t,x,u,\mu) = (1-\mu) f(t,x,u) + \mu \widehat{f}(x,u).$$

Observe that the mapping h is a Carathéodory function (see condition $(\widetilde{C})$ on p. 53) satisfying conditions (h1) and (h2) (see p. 53) with the same functions l and m_0 as the map f. Consequently, by Corollary 3.1.5, the Nemytskii operator $H : [0,+\infty) \times H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ associated with the function h, is well-defined, continuous, fulfills conditions (**H1**) and (**H2**) (see p. 54). As l is such that (5.7) holds, from Corollary 3.3.4 we deduce that

$$\chi_{L^2}(H(D \times U \times [0,1])) \leq \widehat{\varrho}(l_\infty)\chi_{L^2}(U) < \sqrt{d}\,\rho\,\chi_{L^2}(U), \tag{5.13}$$

for any bounded $D \subset [0,+\infty)$ and $U \subset H^1(\mathbb{R}^N)$. Moreover, for $t \geq 0$, $u \in H^1(\mathbb{R}^N)$ and $\mu \in [0,1]$, we have

$$H(t,u,\mu) = (1-\mu)F(t,u) + \mu \widehat{F}(u).$$

Next, let $\Psi^{(\varepsilon)}_t : \mathbb{X} \times [0,1] \to \mathbb{X}$ be the translation along trajectories operator associated with the equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t),v(t)) + \big(0, H(t/\varepsilon, u(t), \mu)\big), \quad t > 0, \tag{5.14}$$

where $\varepsilon \in (0,1]$ is a parameter. Observe that, for any fixed $\varepsilon \in (0,1]$, $H(\cdot/\varepsilon,\cdot,\cdot) : [0,+\infty) \times H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ is continuous, satisfies (**H1**) and (**H2**), and fulfills (5.13), for any bounded $D \subset [0,+\infty)$ and $U \subset H^1(\mathbb{R}^N)$. Therefore, Theorem 3.4.1 shows that, for any $t > 0$ and $\varepsilon \in (0,1]$, the translation operator $\Psi^{(\varepsilon)}_t$ is a well-defined, continuous k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$.

We claim that there exists $\varepsilon_0 \in (0,1]$ such that

$$\Psi^{(\varepsilon)}_{\varepsilon T}(\overline{u},\overline{v},\mu) \neq (\overline{u},\overline{v}) \ \text{ for all } \varepsilon \in (0,\varepsilon_0],\ (\overline{u},\overline{v}) \in \partial Z, \text{ and } \mu \in [0,1]. \tag{5.15}$$

Indeed, suppose to the contrary that (5.15) is not true. Then, there exist sequences $(\varepsilon_n)_{n\geq 1}$ in $(0,1]$, $(\mu_n)_{n\geq 1}$ in $[0,1]$, and $\big((\overline{u}_n,\overline{v}_n)\big)_{n\geq 1}$ in ∂Z such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\Psi^{(\varepsilon_n)}_{\varepsilon_n T}(\overline{u}_n,\overline{v}_n,\mu_n) = (\overline{u}_n,\overline{v}_n)$ for $n \geq 1$. Hence, for any $n \geq 1$, there exists the $\varepsilon_n T$-periodic mild solution $(u_n,v_n) : [0,+\infty) \to \mathbb{X}$ of (5.14) with $\varepsilon = \varepsilon_n$, $\mu = \mu_n$, and $(u_n(0),v_n(0)) = (\overline{u}_n,\overline{v}_n)$. Without loss of generality, we may assume that $\mu_n \to \mu_0$ for some $\mu_0 \in [0,1]$. Let us define, for any $n \geq 0$, the T-periodic mapping $f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as $f_n(t,x,u) = h(t,x,u,\mu_n)$. Since h is a Carathéodory function satisfying (h1) and (h2) with the same functions l and m_0 as f, we get that f_n, $n \geq 0$, are also Carathéodory functions satisfying (f1) and (f2) with the same functions l and m_0. As $h(\cdot,x,\cdot,\cdot) : [0,+\infty) \times \mathbb{R} \times [0,1] \to \mathbb{R}$ is continuous, we see that $f_n \to f_0$ in the sense of condition (f0). Let $F_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \geq 0$, be the Nemytskii operators associated with functions f_n, $n \geq 0$. Observe that, for any $n \geq 1$, $t \geq 0$ and $u \in H^1(\mathbb{R}^N)$,

$$F_n(t/\varepsilon_n, u) = H(t/\varepsilon_n, u, \mu_n).$$

Hence, by Proposition 5.1.2 (ii), we may assume that $(\overline{u}_n, \overline{v}_n) \to (\overline{u}_0, \overline{v}_0)$ as $n \to \infty$, for $(\overline{u}_0, \overline{v}_0) \in D(\mathbb{A}) \cap \partial Z$ such that

$$\mathbb{A}(\overline{u}_0, \overline{v}_0) + (0, \widehat{F}_0(\overline{u}_0)) = 0, \tag{5.16}$$

where $\widehat{F}_0 : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the function $\widehat{f}_0$. Since, in view of Proposition 3.1.2 (ii), $\widehat{F}_0$ is the time-averaged operator of F_0, we see finally that, for any $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\widehat{F}_0(u) &= \frac{1}{T}\int_0^T F_0(t,u)\,dt = \frac{1}{T}\int_0^T H(t,u,\mu_0)\,dt \\ &= \frac{1}{T}\int_0^T (1-\mu_0)F(t,u) + \mu_0\widehat{F}(u)\,dt = \widehat{F}(u).\end{aligned}$$

Hence, (5.16) contradicts (5.11), which proves (5.15), as desired. Thus, for any fixed $\varepsilon \in (0, \varepsilon_0]$, the homotopy invariance of the fixed-point index (see Theorem A.4.8) yields

$$\mathrm{Ind}_C(\Phi^{(\varepsilon)}_{\varepsilon T}, Z) = \mathrm{Ind}_C(\Psi^{(\varepsilon)}_{\varepsilon T}(\cdot, 0), Z) = \mathrm{Ind}_C(\Psi^{(\varepsilon)}_{\varepsilon T}(\cdot, 1), Z) = \mathrm{Ind}_C(\widehat{\Phi}_{\varepsilon T}, Z).$$

Therefore, we obtain the index formula (5.12), which completes the proof. □

Using the above index averaging formula, we get the following *continuation principle.*

Theorem 5.2.2. *Suppose that $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $Z \subset \mathbb{X}$ are as in Theorem 5.2.1. If, for any $\varepsilon \in (0, 1]$, the equation* (5.3) *has no εT-periodic mild solutions with $(u(0), v(0)) \in \partial Z$, then*

$$\mathrm{Ind}_C(\Phi_T, Z) = \lim_{t \to 0^+} \mathrm{Ind}_C(\widehat{\Phi}_t, Z)$$

where Φ_t is the translation operator associated with the equation (5.2).

Proof. In view of the assumptions, for all $\varepsilon \in (0, 1]$ and $(\overline{u}, \overline{v}) \in \partial Z$, we have

$$\Phi^{(\varepsilon)}_{\varepsilon T}(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}). \tag{5.17}$$

Consequently, from Theorem 5.2.1 it follows that there exists $\varepsilon_0 \in (0, 1]$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ and $(\overline{u}, \overline{v}) \in \partial Z$,

$$\widehat{\Phi}_{\varepsilon T}(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \ \text{ and } \ \mathrm{Ind}_C(\Phi^{(\varepsilon)}_{\varepsilon T}, Z) = \mathrm{Ind}_C(\widehat{\Phi}_{\varepsilon T}, Z). \tag{5.18}$$

Given $\varepsilon_1 \in (0, \varepsilon_0]$, we consider the translation operator $\widetilde{\Phi}_t : \mathbb{X} \times [\varepsilon_1, 1] \to \mathbb{X}$ associated with the equation

$$(\dot{u}(t), \dot{v}(t)) = \varepsilon\mathbb{A}(u(t), v(t)) + \varepsilon\big(0, F(t, u(t))\big), \quad t > 0. \tag{5.19}$$

Theorem 3.4.1 implies that, for any $t > 0$, $\widetilde{\Phi}_t$ is a well-defined, continuous k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$. Then, changing variables in the Duhamel formula, we obtain, for all $t > 0$, all $\varepsilon \in [\varepsilon_1, 1]$, and all $(\overline{u}, \overline{v}) \in \mathbb{X}$,

$$\widetilde{\Phi}_t(\overline{u}, \overline{v}, \varepsilon) = \Phi^{(\varepsilon)}_{\varepsilon t}(\overline{u}, \overline{v}).$$

Hence, in particular, $\Phi_t = \widetilde{\Phi}_t(\cdot, 1)$. This and (5.17) entail that $\widetilde{\Phi}_T$ is the admissible homotopy on the closure of the set Z, thus, in view of the homotopy invariance of the fixed-point index (see Theorem A.4.8) and (5.18),

$$\mathrm{Ind}_C(\Phi_T, Z) = \mathrm{Ind}_C(\widetilde{\Phi}_T(\cdot, 1), Z) = \mathrm{Ind}_C(\widetilde{\Phi}_T(\cdot, \varepsilon_1), Z) = \mathrm{Ind}_C(\Phi^{(\varepsilon_1)}_{\varepsilon_1 T}, Z) = \mathrm{Ind}_C(\widehat{\Phi}_{\varepsilon_1 T}, Z).$$

Since $\varepsilon_1 \in (0, \varepsilon_0]$ was arbitrary, the proof is completed. □

5.3 *A priori* estimates

In order to apply the continuation principle (Theorem 5.2.2) we will need *a priori* estimates. To this end, we recall notions of linearizations and non-resonance conditions.

Namely, we say that function $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *admits a linearization at infinity* (or it is *asymptotically linear at infinity*, see (10)), if there exists a Carathéodory function $\omega : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{|u|\to\infty} \frac{f(t,x,u)}{u} = \omega(t,x) \tag{5.20}$$

for almost every $x \in \mathbb{R}^N$, uniformly on bounded subsets of $[0,+\infty)$ with respect to t. We then say that the function ω is the *coefficient of the linearization at infinity* (or it is the *asymptotic coefficient at infinity*), and the function $f^\omega : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f^\omega(t,x,u) = \omega(t,x)u \tag{5.21}$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all $u \in \mathbb{R}$, is the *linearization of f at infinity.* Moreover, we say that f *admits a linearization at zero* (or it is *asymptotically linear at zero*, see (11)), if there exists a Carathéodory function $\alpha : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{u\to 0} \frac{f(t,x,u)}{u} = \alpha(t,x) \tag{5.22}$$

for almost every $x \in \mathbb{R}^N$, uniformly on bounded subsets of $[0,+\infty)$ with respect to t. As before, the function α is called the *coefficient of the linearization at zero* (or the *asymptotic coefficient at zero*), and the function $f^\alpha : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$f^\alpha(t,x,u) = \alpha(t,x)u \tag{5.23}$$

for almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all $u \in \mathbb{R}$, is called the *linearization of f at zero.*

We shall assume that (cf. (12))

$$\widehat{\omega}(x) = \frac{1}{T}\int_0^T \omega(t,x)\,dt \quad \text{and} \quad \widehat{\alpha}(x) = \frac{1}{T}\int_0^T \alpha(t,x)\,dt, \tag{5.24}$$

for almost every $x \in \mathbb{R}^N$, are Kato-Rellich type functions (see (8a) and (8b)). We shall also assume the *non-resonance condition at infinity* (see (15)), that is,

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}) = \{0\}, \tag{5.25}$$

and the *non-resonance condition at zero* (see (15)), that is,

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}) = \{0\} \tag{5.26}$$

where the linear operators $\widehat{\boldsymbol{\omega}}$, $\widehat{\boldsymbol{\alpha}} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are given by

$$[\widehat{\boldsymbol{\omega}}u](x) = \widehat{\omega}(x)u(x) \quad \text{and} \quad [\widehat{\boldsymbol{\alpha}}u](x) = \widehat{\alpha}(x)u(x), \tag{5.27}$$

for all $u \in H^1(\mathbb{R}^N)$, and almost every $x \in \mathbb{R}^N$.

Now, we state the result concerning the *a priori* estimates.

Proposition 5.3.1. *Assume that $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a T-periodic Carathéodory function satisfying* (f1) *and* (f2)*, and function l from* (f1) *is such that* (5.7) *holds.*

(i) *Suppose that f admits a linearization at infinity with a coefficient of the linearization $\omega : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $\widehat{\omega}$ defined by* (5.24) *is a Kato-Rellich type function. In addition, assume that the non-resonance condition at infinity* (5.25) *is satisfied. If, for any $\varepsilon \in (0,1]$, the linearized equation*

$$u_{tt} + \beta u_t = \Delta u - V(x)u + \omega(t/\varepsilon, x)u, \quad t > 0, \ x \in \mathbb{R}^N, \tag{5.28}$$

does not admit nonzero εT-periodic mild solutions, then there exists $R_0 > 0$ such that, for any $\varepsilon \in (0,1]$, the equation (5.3) *does not admit εT-periodic mild solutions such that $\|(u(0), v(0))\|_{\mathbb{X}} \geq R_0$.*

(ii) *Suppose that f admits a linearization at zero with a coefficient of the linearization $\alpha : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $\widehat{\alpha}$ defined by* (5.24) *is a Kato-Rellich type function. In addition, assume that the non-resonance condition at zero* (5.26) *is satisfied. If, for any $\varepsilon \in (0,1]$, the linearized equation*

$$u_{tt} + \beta u_t = \Delta u - V(x)u + \alpha(t/\varepsilon, x)u, \quad t > 0, \ x \in \mathbb{R}^N, \tag{5.29}$$

does not admit nonzero εT-periodic mild solutions, then there exists $r_0 > 0$ such that, for any $\varepsilon \in (0,1]$, the equation (5.3) *does not admit εT-periodic mild solutions such that $0 < \|(u(0), v(0))\|_{\mathbb{X}} \leq r_0$.*

Before the proof we provide the following remark.

Remark 5.3.2. Let $f : [0,+\infty)\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ be a T-periodic Carathéodory function satisfying (f1) with l and (f2) with m_0. Moreover, suppose that f admits a linearization at infinity and at zero with coefficients of linearization ω, $\alpha : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$, and linearizations f^ω, $f^\alpha : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, respectively.

(i) The existence of a linearization at zero implies that $f(t,x,0) = 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$.

(ii) The coefficients of linearizations ω and α are T-periodic, and fulfill $|\omega(t,x)| \leq l(x)$ and $|\alpha(t,x)| \leq l(x)$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. In fact, the periodicity follows from the periodicity of f. Further, if f admits a linearization at infinity, then by (f1) and (f2), we get, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$,

$$\left| \frac{f(t,x,u)}{u} \right| \leq \frac{l(x)|u| + m_0(x)}{|u|} \to l(x) \ \text{ as } |u| \to \infty.$$

Next, if there exists a linearization at zero, then by (f1), one obtains that, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$,

$$\left| \frac{f(t,x,u)}{u} \right| \leq \frac{l(x)|u| + |f(t,x,0)|}{|u|} = l(x),$$

because, by (i), $f(t,x,0) = 0$ for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$.

(iii) The linearizations f^ω and f^α defined by (5.21) and by (5.23), respectively, are T-periodic Carathéodory functions satisfying condition (f1) with the same function l as the mapping f, and condition (f2) with the function $m_0 \equiv 0$. Indeed, by the assumption, ω and α are Carathéodory functions and, by the point (ii), they are T-periodic, and $|\omega(t,x)| \leq l(x)$ and $|\alpha(t,x)| \leq l(x)$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. Therefore, the mappings f^ω, f^α are as desired. Let F^ω, $F^\alpha : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, be the Nemytskii operators associated with functions f^ω, f^α, respectively. It follows from Proposition 3.1.2 (i) that the mappings F^ω and F^α are well-defined, continuous, satisfy condition (**F1**) with a constant $\|L\|_{\mathcal{L}(H^1,L^2)}$, where $L : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the linear operator of multiplication by a function l (see Corollary 3.1.1), and condition (**F2**) with $\|m_0\|_{L^2} = 0$. Therefore, Theorem 3.4.1 (i) implies that, for any $\varepsilon \in (0,1]$ and $(\overline{u}, \overline{v}) \in \mathbb{X}$, there exists the unique mild solution $(u,v) : [0,+\infty) \to \mathbb{X}$ of the linearized equation (5.28) (respectively, of the equation (5.29)) with the initial condition $(u(0), v(0)) = (\overline{u}, \overline{v})$.

(iv) The linear operators $\widehat{\boldsymbol{\omega}}$ and $\widehat{\boldsymbol{\alpha}} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, defined by (5.27), are well-defined and bounded. Additionally, for any $u \in H^1(\mathbb{R}^N)$,

$$\widehat{\boldsymbol{\omega}}\, u = \widehat{F}^{\omega}(u) \quad \text{and} \quad \widehat{\boldsymbol{\alpha}}\, u = \widehat{F}^{\alpha}(u),$$

where $\widehat{F}^{\omega}$, $\widehat{F}^{\alpha} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are the Nemytskii operators associated with the time-averaged functions $\widehat{f}^{\omega}$ and $\widehat{f}^{\alpha} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{f}^{\omega}(x,u) = \frac{1}{T}\int_0^T f^{\omega}(t,x,u)\,dt \quad \text{and} \quad f^{\alpha}(x,u) = \frac{1}{T}\int_0^T f^{\alpha}(t,x,u)\,dt,$$

for almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$.

In fact, by the assumption, $\widehat{\omega}$ and $\widehat{\alpha}$, given by (5.24), are Kato-Rellich type functions. Using Corollary 3.1.1 we obtain that the linear operators $\widehat{\boldsymbol{\omega}}$ and $\widehat{\boldsymbol{\alpha}}$ are well-defined and bounded. In view of the point (iii), the mappings f^{ω} and f^{α}, defined by (5.21) and (5.23), respectively, are T-periodic Carathéodory functions satisfying condition (f1) with the same function l as f, and condition (f2) with $m_0 \equiv 0$. Thus, by Proposition 3.1.2 (ii), $\widehat{f}^{\omega}$ and $\widehat{f}^{\alpha}$ are Carathéodory functions satisfying (f1) with the same l as f, and (f2) with $m_0 \equiv 0$, and the mappings $\widehat{F}^{\omega}$, $\widehat{F}^{\alpha}$ are well-defined. Applying directly the definitions of the linear operator $\widehat{\boldsymbol{\omega}}$ and the Nemytskii operator $\widehat{F}^{\omega}$, we obtain, for all $u \in H^1(\mathbb{R}^N)$ and almost every $x \in \mathbb{R}^N$,

$$\begin{aligned}[\widehat{\boldsymbol{\omega}}\, u](x) &= \widehat{\omega}(x)u(x) = \frac{1}{T}\int_0^T \omega(t,x)u(x)\,dt \\ &= \frac{1}{T}\int_0^T f^{\omega}(t,x,u(x))\,dt = \widehat{f}^{\omega}(x,u(x)) = [\widehat{F}^{\omega}(u)](x),\end{aligned}$$

hence $\widehat{\boldsymbol{\omega}}\, u = \widehat{F}^{\omega}(u)$, as claimed. Similarly we get that $\widehat{\boldsymbol{\alpha}}\, u = \widehat{F}^{\alpha}(u)$ for all $u \in H^1(\mathbb{R}^N)$.

□

We will need the following lemma.

Lemma 5.3.3. *Assume that* $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is a* T*-periodic Carathéodory function satisfying* (f1) *and* (f2)*, and* $\mu_n \to \mu_0$ *in* $[0,1]$.

(i) *Suppose that* f *admits a linearization at infinity,* $\omega : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ *is the coefficient of the linearization, and* $(\rho_n)_{n\geq 1}$ *is a sequence in* $(0,+\infty)$ *such that* $\rho_n \to +\infty$ *as* $n \to \infty$. *Consider functions* $f_n^{\omega} : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, *given by*

$$f_n^{\omega}(t,x,u) = (1-\mu_n)\rho_n^{-1} f(t,x,\rho_n u) + \mu_n \omega(t,x)u.$$

Then, f_n^{ω}, $n \geq 1$, *are* T*-periodic Carathéodory functions satisfying condition* (f1) *with the same function* l *as* f*, and condition* (f2) *with the function* Cm_0*, where the mapping* m_0 *comes from the condition* (f2) *for* f*, and* $C > 0$ *is such that* $\rho_n^{-1} \leq C$ *for* $n \geq 1$*. Moreover,* $f_n^{\omega} \to f^{\omega}$ *in the sense of condition* (f0)*, where the mapping* f^{ω} *is defined by* (5.21).

(ii) *Suppose that f admits a linearization at zero, $\alpha : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ is the coefficient of the linearization, and $(\rho_n)_{n\geq 1}$ is a sequence in $(0,+\infty)$ such that $\rho_n \to 0$ as $n \to \infty$. Consider functions $f_n^\alpha : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, given by*

$$f_n^\alpha(t,x,u) = (1-\mu_n)\rho_n^{-1} f(t,x,\rho_n u) + \mu_n \alpha(t,x)u. \tag{5.30}$$

Then, f_n^α, $n \geq 1$, are T-periodic Carathéodory functions satisfying condition (f1) *with the same function l as f, and condition* (f2) *with the function $m_0 \equiv 0$. Moreover, $f_n^\alpha \to f^\alpha$ in the sense of condition* (f0)*, where the mapping f^α is defined by* (5.23).

Proof. (i) Let $g_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, be given by

$$g_n(t,x,u) = \rho_n^{-1} f(t,x,\rho_n u). \tag{5.31}$$

Since f is a T-periodic Carathéodory function, so are the mappings g_n for $n \geq 1$. Using the condition (f1) for f we get that, for $n \geq 1$, almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all u_1, $u_2 \in \mathbb{R}$,

$$|g_n(t,x,u_1) - g_n(t,x,u_2)| \leq \rho_n^{-1} l(x)|\rho_n u_1 - \rho_n u_2| = l(x)|u_1 - u_2|. \tag{5.32}$$

Thus, g_n, $n \geq 1$, satisfy the condition (f1) with the same function l as f. As $\rho_n \to +\infty$, there exists $C > 0$ such that $\rho_n^{-1} \leq C$ for $n \geq 1$. Hence, g_n, $n \geq 1$, fulfill the condition (f2) with the function Cm_0, where the mapping m_0 comes from the condition (f2) for f.

Observe that

$$f_n^\omega(t,x,u) = (1-\mu_n) g_n(t,x,u) + \mu_n f^\omega(t,x,u) \tag{5.33}$$

are T-periodic Carathéodory functions, because, by Remark 5.3.2 (iii), f^ω is a T-periodic Carathéodory function. Moreover, it follows from Remark 5.3.2 (iii) that f^ω satisfies the condition (f1) with the same function l as f. Hence, for any $n \geq 1$, almost every $x \in \mathbb{R}^N$, all $t \geq 0$, and all u_1, $u_2 \in \mathbb{R}$,

$$\begin{aligned} |f_n^\omega(t,x,u_1) - f_n^\omega(t,x,u_2)| &\leq (1-\mu_n)|g_n(t,x,u_1) - g_n(t,x,u_2)| \\ &\quad + \mu_n |f^\omega(t,x,u_1) - f^\omega(t,x,u_2)| \\ &\leq l(x)|u_1 - u_2|. \end{aligned} \tag{5.34}$$

Hence, f_n^ω, $n \geq 1$, satisfy (f1) with the same l as f. Note that, for any $n \geq 1$, almost every $x \in \mathbb{R}^N$, and all $t \geq 0$,

$$|f_n^\omega(t,x,0)| \leq (1-\mu_n)|g_n(t,x,0)| \leq Cm_0. \tag{5.35}$$

This means that the mappings f_n^ω, $n \geq 1$, satisfy (f2) with the function Cm_0.

By the assumption f admits a linearization at infinity, hence, the limit (5.20) exists, for almost every $x \in \mathbb{R}^N$, and uniformly on bounded subsets of $[0,+\infty)$ with respect to t. Let $x \in \mathbb{R}^N$ be such that the limit (5.20) exists and the condition (f2) for f is satisfied. If $u \neq 0$, then $|\rho_n u| \to +\infty$ as $n \to \infty$ and

$$g_n(t,x,u) = \rho_n^{-1} f(t,x,\rho_n u) = (\rho_n u)^{-1} f(t,x,\rho_n u) u \xrightarrow{n\to\infty} \omega(t,x)u = f^\omega(t,x,u), \tag{5.36}$$

uniformly on bounded subsets of $[0,+\infty)$ with respect to t. If $u = 0$, then, by the condition (f2) for f, one obtains $g_n(t,x,0) \to 0 = f^\omega(t,x,0)$ as $n \to \infty$, uniformly on bounded subsets of $[0,+\infty)$ with respect to t. Hence, we see that $g_n \to f^\omega$ in the sense of condition (f0). Consequently, we obtain

$$\begin{aligned} f_n^\omega(t,x,u) &= (1-\mu_n) g_n(t,x,u) + \mu_n f^\omega(t,x,u) \\ &\xrightarrow{n\to\infty} (1-\mu_0) f^\omega(t,x,u) + \mu_0 f^\omega(t,x,u) = f^\omega(t,x,u), \end{aligned} \tag{5.37}$$

for almost every $x \in \mathbb{R}^N$, all $u \in \mathbb{R}$, and uniformly on bounded subsets of $[0,+\infty)$ with respect to t, i.e., $f_n^\omega \to f^\omega$ in the sense of condition (f0).

(ii) Let g_n, $n \geq 1$, be defined by (5.31), where $\rho_n \to 0$. As in the proof of (i), we can show that g_n, $n \geq 1$, are T-periodic Carathéodory functions. In view of (5.32), g_n, $n \geq 1$, satisfy (f1) with the same l as f. Furthermore, by Remark 5.3.2 (i), $g_n(t,x,0) = 0$, for $n \geq 1$, almost every $x \in \mathbb{R}^N$, and all $t \geq 0$. Hence, the mappings g_n, $n \geq 1$, satisfy (f2) with the function $m_0 \equiv 0$.

Observe that

$$f_n^\alpha(t,x,u) = g_n(t,x,u) + f^\alpha(t,x,u), \quad n \geq 1,$$

are T-periodic Carathéodory functions, because, by Remark 5.3.2 (iii), f^α is a T-periodic Carathéodory function. Moreover, it follows from Remark 5.3.2 (iii) that f^α satisfies the condition (f1) with the same function l as f. Therefore, from (5.34) we deduce that the mappings f_n, $n \geq 1$, satisfy (f1) with the same l as f. In addition, we see that functions f_n, $n \geq 1$, satisfy (f2) with $m_0 \equiv 0$.

By the assumption, f admits a linearization at zero, hence, the limit (5.22) exists, for almost every $x \in \mathbb{R}^N$, uniformly for t on bounded subsets of $[0,+\infty)$. Let $x \in \mathbb{R}^N$ be such that the limit (5.22) exists and $f(t,x,0) = 0$ for all $t \geq 0$. If $u \neq 0$, then, similarly as in (5.36), $g_n(t,x,u) \xrightarrow{n\to\infty} f^\alpha(t,x,u)$, uniformly for t on bounded subsets of $[0,+\infty)$. If $u = 0$, then $g_n(t,x,0) = 0$, for $n \geq 1$ and $t \geq 0$, hence again $g_n(t,x,0) \to 0 = f^\alpha(t,x,0)$ as $n \to \infty$, uniformly for t on bounded subsets of $[0,+\infty)$. Thus, $g_n \to f^\alpha$ in the sense of condition (f0). Arguing analogously to (5.37), we also obtain that $f_n \to f^\alpha$ in the sense of (f0). Hence, the proof is completed. □

Proof of Proposition 5.3.1. (i) Suppose, to the contrary, that there exist sequences $(\varepsilon_n)_{n\geq 1}$ in $(0,1]$ and $\big((\overline{u}_n,\overline{v}_n)\big)_{n\geq 1}$ in $\mathbb{X}$ such that $\rho_n = \|(\overline{u}_n,\overline{v}_n)\|_{\mathbb{X}} \xrightarrow{n\to\infty} +\infty$ and, for any $n \geq 1$, there exists the $\varepsilon_n T$-periodic mild solution $(u_n,v_n) : [0,+\infty) \to \mathbb{X}$ of the problem (5.3) with $\varepsilon = \varepsilon_n$ and the initial condition $(u_n(0),v_n(0)) = (\overline{u}_n,\overline{v}_n)$. Next, we define functions $(\widetilde{u}_n,\widetilde{v}_n) : [0,+\infty) \to \mathbb{X}$, $n \geq 1$, by

$$(\widetilde{u}_n(t),\widetilde{v}_n(t)) = \rho_n^{-1}(u_n(t),v_n(t)) \ \text{ for } t \geq 0. \tag{5.38}$$

Using the Duhamel formula, we easily see that functions $(\widetilde{u}_n,\widetilde{v}_n)$, $n \geq 1$, are the $\varepsilon_n T$-periodic mild solutions of the Cauchy problem

$$\begin{cases} (\dot{u}(t),\dot{v}(t)) = \mathbb{A}(u(t),v(t)) + \big(0,\rho_n^{-1}F(t/\varepsilon_n,\rho_n u(t))\big), & t > 0, \\ (u(0),v(0)) = \rho_n^{-1}(\overline{u}_n,\overline{v}_n). \end{cases} \tag{5.39}$$

Let $f_n : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, be mappings defined by

$$f_n(t,x,u) = \rho_n^{-1} f(t,x,\rho_n u). \tag{5.40}$$

From Lemma 5.3.3 (i) applied for the sequence $\mu_n \equiv 0$, we deduce that f_n, $n \geq 1$, are T-periodic Carathéodory functions satisfying condition (f1) with the same function l as the mapping f, and condition (f2) with function Cm_0, where the mapping m_0 comes from the condition (f2) for f, and $C > 0$ is such that $\rho_n^{-1} \leq C$ for all $n \geq 1$. It follows from Remark 5.3.2 (iii) that function f^ω given by (5.21) is also a T-periodic Carathéodory function satisfying condition (f1) with the same l as the mapping f and condition (f2) with the function $m_0 \equiv 0$. Furthermore, again by Lemma 5.3.3 (i), $f_n \to f^\omega$ in the sense of condition (f0). Let $F_n : [0,+\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$, $n \geq 0$, be the Nemytskii operators associated with functions f_n, $n \geq 1$. We see that $F_n(t,u) = \rho_n^{-1}F(t,\rho_n u)$ for all $n \geq 1$, $t \geq 0$, and $u \in H^1(\mathbb{R}^N)$, hence, for any $n \geq 1$, function $(\widetilde{u}_n,\widetilde{v}_n)$ is the $\varepsilon_n T$-periodic mild solution of the problem (5.3) with $F = F_n$, $\varepsilon = \varepsilon_n$, and the initial condition $(\widetilde{u}_n(0),\widetilde{v}_n(0)) = \rho_n^{-1}(\overline{u}_n,\overline{v}_n)$. Without loss of generality, we may assume that $\varepsilon_n \to \varepsilon_0$, where $\varepsilon_0 \in [0,1]$.

Suppose that $\varepsilon_0 > 0$. Since functions f_n, $n \geq 1$, and f^ω satisfy condition (f1) with the function l, which is assumed to be such that (5.7) holds, it follows from Proposition 5.1.2 (i) that, passing to a subsequence if necessary,

$$\rho_n^{-1}(\overline{u}_n,\overline{v}_n) \to (\varphi,\xi) \quad \text{as } n \to \infty$$

for some $(\varphi, \xi) \in \mathbb{X}$. Recall that $F^\omega : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined as the Nemytskii operator associated with the function f^ω (see Remark 5.3.2 (iii)). Let $(\widetilde{u}_0, \widetilde{v}_0) : [0, +\infty) \to \mathbb{X}$ be the mild solution of the equation (5.3) with $F = F^\omega$, $\varepsilon = \varepsilon_0$, and the initial condition $(\widetilde{u}_0(0), \widetilde{v}_0(0)) = (\varphi, \xi)$. Again by Proposition 5.1.2 (i), the limit

$$(\widetilde{u}_n(t), \widetilde{v}_n(t)) \to (\widetilde{u}_0(t), \widetilde{v}_0(t)) \quad \text{as } n \to \infty \tag{5.41}$$

holds uniformly on bounded subsets of $[0, +\infty)$ with respect to t, and the solution $(\widetilde{u}_0, \widetilde{v}_0)$ is $\varepsilon_0 T$-periodic. Observe that $[F^\omega(t, u)](x) = \omega(t, x)u(x)$ for all $t \geq 0$, $u \in H^1(\mathbb{R}^N)$, and almost every $x \in \mathbb{R}^N$, hence $(\widetilde{u}_0, \widetilde{v}_0)$ is the mild solution of (5.28). Furthermore, as, for any $n \geq 1$, $\|\rho_n^{-1}(\overline{u}_n, \overline{v}_n)\|_{\mathbb{X}}=1$, we have $\|(\widetilde{u}_0(0), \widetilde{v}_0(0))\|_{\mathbb{X}} = \|(\varphi, \xi)\|_{\mathbb{X}} = 1$. Therefore, $(\widetilde{u}_0, \widetilde{v}_0)$ is the nonzero $\varepsilon_0 T$-periodic mild solution of (5.28), which contradicts the assumption.

Now, suppose that $\varepsilon_0 = 0$. In view of Proposition 5.1.2 (ii), we may assume, extracting a subsequence if necessary, that $\rho_n^{-1}(\overline{u}_n, \overline{v}_n) \to (\varphi, \xi) \in D(\mathbb{A})$, and

$$\mathbb{A}(\varphi, \xi) + \big(0, \widehat{F}^\omega(\varphi)\big) = 0$$

where $\widehat{F}^\omega : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the time-averaged function $\widehat{f}^\omega$. By Remark 5.3.2 (iv), $\widehat{F}^\omega(u) = \widehat{\boldsymbol{\omega}}\, u$ for all $u \in H^1(\mathbb{R}^N)$, where the linear operator $\widehat{\boldsymbol{\omega}} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined by (5.27). Therefore, it follows from the equation above that $\xi = 0$ and

$$0 = -(-\Delta + \boldsymbol{V})\varphi + \widehat{F}^\omega(\varphi) = -(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}})\varphi.$$

Moreover, $\|(\varphi, \xi)\|_{\mathbb{X}} = 1$, hence $\varphi \neq 0$. As a result, we have reached a contradiction with the non-resonance condition at infinity (5.25). This proves that there exists $R_0 > 0$ such that, for any $\varepsilon \in (0, 1]$, the equation (5.3) has no εT-periodic mild solutions (u, v) satisfying $\|(u(0), v(0))\|_{\mathbb{X}} \geq R_0$.

(ii) Suppose, to the contrary, that there are sequences $(\varepsilon_n)_{n\geq 1}$ in $(0, 1]$ and $\big((\overline{u}_n, \overline{v}_n)\big)_{n\geq 1}$ in $\mathbb{X}$, such that $\rho_n = \|(\overline{u}_n, \overline{v}_n)\|_{\mathbb{X}} \xrightarrow{n\to\infty} 0$ and, for any $n \geq 1$, there exists the $\varepsilon_n T$-periodic mild solution $(u_n, v_n) : [0, +\infty) \to \mathbb{X}$ of the equation (5.3) with $\varepsilon = \varepsilon_n$ and the initial condition $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$. Next, we define functions $(\widetilde{u}_n, \widetilde{v}_n) : [0, +\infty) \to \mathbb{X}$, $n \geq 1$, by the formula (5.38). As before, $(\widetilde{u}_n, \widetilde{v}_n)$, $n \geq 1$, are the $\varepsilon_n T$-periodic mild solutions of the Cauchy problem (5.39). For any $n \geq 1$ let f_n be a function given by the formula (5.40). By Lemma 5.3.3 (ii) applied for the sequence $\mu_n \equiv 0$, f_n, $n \geq 1$, are T-periodic Carathéodory functions satisfying condition (f1) with the same function l as the mapping f, and condition (f2) with the function $m_0 \equiv 0$. It follows from Remark 5.3.2 (iii) that function f^α defined by (5.23) is also a T-periodic Carathéodory function satisfying condition (f1) with the same l as the mapping f and condition (f2) with the function $m_0 \equiv 0$. Applying again Lemma 5.3.3 (ii), we get that $f_n \to f^\alpha$ in the sense of condition (f0). As before, let F_n be the Nemytskii operators associated with the functions f_n for $n \geq 1$. Observe that $F_n(t, u) = \rho_n^{-1}F(t, \rho_n u)$ for all $n \geq 1$, $t \geq 0$, and $u \in H^1(\mathbb{R}^N)$, hence, for any $n \geq 1$, $(\widetilde{u}_n, \widetilde{v}_n)$ is the $\varepsilon_n T$-periodic mild solution of the equation (5.3) with $F = F_n$, $\varepsilon = \varepsilon_n$, and the initial condition $(\widetilde{u}_n(0), \widetilde{v}_n(0)) = \rho_n^{-1}(\overline{u}_n, \overline{v}_n)$. Without loss of generality, we may assume that $\varepsilon_n \to \varepsilon_0$, where $\varepsilon_0 \in [0, 1]$.

Suppose that $\varepsilon_0 > 0$. We apply Proposition 5.1.2 (i) to get, passing to a subsequence if necessary, that $\rho_n^{-1}(\overline{u}_n, \overline{v}_n) \xrightarrow{n\to\infty} (\varphi, \xi)$ for a certain $(\varphi, \xi) \in \mathbb{X}$. Recall that $F^\alpha : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined as the Nemytskii operator associated with the function f^α (see Remark 5.3.2 (iii)). Let $(\widetilde{u}_0, \widetilde{v}_0) : [0, +\infty) \to \mathbb{X}$ be the mild solution of the equation (5.3) with $F = F^\alpha$, $\varepsilon = \varepsilon_0$, and the initial condition $(\widetilde{u}_0(0), \widetilde{v}_0(0)) = (\varphi, \xi)$. Using Proposition 5.1.2 (i) once more, we obtain that the convergence

$$(\widetilde{u}_n(t), \widetilde{v}_n(t)) \to (\widetilde{u}_0(t), \widetilde{v}_0(t)) \quad \text{as } n \to \infty$$

is uniform on bounded subsets of $[0, +\infty)$ with respect to t, and the solution $(\widetilde{u}_0, \widetilde{v}_0)$ is $\varepsilon_0 T$-periodic. Observe that $[F^\alpha(t, u)](x) = \alpha(t, x)u(x)$ for all $t \geq 0$, all $u \in H^1(\mathbb{R}^N)$, and almost every $x \in \mathbb{R}^N$, hence $(\widetilde{u}_0, \widetilde{v}_0)$ is the mild solution of the equation (5.29). Since $\|(\widetilde{u}_0(0), \widetilde{v}_0(0))\|_{\mathbb{X}} =$

$\|(\varphi,\xi)\|_{\mathbb{X}} = 1$, we get that $(\widetilde{u}_0, \widetilde{v}_0)$ is the nonzero $\varepsilon_0 T$-periodic mild solution of the equation (5.29), which contradicts the assumption.

Now, suppose that $\varepsilon_0 = 0$. From Proposition 5.1.2 (ii) we deduce that, extracting a subsequence if necessary, that $\rho_n^{-1}(\overline{u}_n, \overline{v}_n) \xrightarrow{n\to\infty} (\varphi,\xi) \in D(\mathbb{A})$, and

$$\mathbb{A}(\varphi,\xi) + \big(0, \widehat{F}^{\alpha}(\varphi)\big)$$

where $\widehat{F}^{\alpha} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the Nemytskii operator associated with the time-averaged function $\widehat{f}^{\alpha}$. It follows from Remark 5.3.2 (iv) that $\widehat{F}^{\alpha}(u) = \widehat{\boldsymbol{\alpha}}\, u$ for all $u \in H^1(\mathbb{R}^N)$, where the linear operator $\widehat{\boldsymbol{\alpha}} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is given by (5.27). Therefore, the equation above yields $\xi = 0$ and

$$0 = -(-\Delta + \boldsymbol{V})\varphi + \widehat{F}^{\alpha}(\varphi) = -(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}})\varphi.$$

Since $\|(\varphi,\xi)\|_{\mathbb{X}} = 1$, one has $\varphi \neq 0$, which contradicts the non-resonance condition at zero (5.26). This shows that there exists $r_0 > 0$ such that, for any $\varepsilon \in (0,1]$, the equation (5.3) has no εT-periodic mild solutions (u,v) satisfying $0 < \|(u(0),v(0))\|_{\mathbb{X}} \le r_0$. The proof is completed. □

If the coefficients of the linearizations are time-independent, the result about *a priori* estimates becomes simpler.

Corollary 5.3.4. *Assume that a T-periodic Carathédory function $f : [0,+\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (f1) *and* (f2)*, and the function l from* (f1) *is such that* (5.7) *holds.*

(i) *Suppose that f admits a linearization at infinity such that the coefficient of the linearization, denoted by ω, is time-independent, and ω is a Kato-Rellich type function satisfying $\varrho(V_\infty - \omega_\infty) > 0$. If the non-resonance condition at infinity* (5.25) *holds, then there exists $R_0 > 0$ such that, for any $\varepsilon \in (0,1]$, the equation* (5.3) *does not admit εT-periodic mild solutions (u,v) with $\|(u(0),v(0))\|_{\mathbb{X}} \ge R_0$.*

(ii) *Suppose that f admits a linearization at zero such that the coefficient of the linearization, denoted by α, is time-independent, and α is a Kato-Rellich type function satisfying $\varrho(V_\infty - \alpha_\infty) > 0$. If the non-resonance condition at zero* (5.26) *holds, then there exists $r_0 > 0$ such that, for any $\varepsilon \in (0,1]$, the equation* (5.3) *does not admit εT-periodic mild solutions (u,v) with $0 < \|(u(0),v(0))\|_{\mathbb{X}} \le r_0$.*

Proof. (i) We shall show that, for any $\varepsilon \in (0,1]$, the linearized equation (5.28) does not admit nonzero εT-periodic mild solutions. To this end, suppose that $(u,v) : [0,+\infty) \to \mathbb{X}$ is an εT-periodic mild solution of the equation (5.28). Since the coefficient of linearization ω is time-independent, the equation (5.28) takes the form

$$u_{tt} + \beta u_t = \Delta u - V(x)u + \omega(x)u, \quad t > 0,\ x \in \mathbb{R}^N.$$

Therefore, the equation above can be rewritten as the following evolution equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + (0, \boldsymbol{\omega}\, u(t)), \quad t > 0,$$

where $\boldsymbol{\omega} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the operator of multiplication by the function ω. Consequently, the equation above becomes

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}^{(\omega)}(u(t), v(t)), \quad t > 0,$$

where the linear operator $\mathbb{A}^{(\omega)} : D(\mathbb{A}^{(\omega)}) \subset \mathbb{X} \to \mathbb{X}$ is given by

$$\mathbb{A}^{(\omega)}(u,v) = \big(v, -(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})u - \beta v\big) \quad \text{for} \quad (u,v) \in D(\mathbb{A}^{(\omega)}) = D(\mathbb{A}).$$

On the other hand, $V - \omega$ is a Kato-Rellich type potential with a positive asymptotic bottom. Hence, applying Lemma 4.2.3, we obtain that there exists $\varphi \in H^2(\mathbb{R}^N)$ such that, for all $t \ge 0$,

$$u(t) = \varphi \in \operatorname{Ker}(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}) \quad \text{and} \quad v(t) = 0.$$

The assumed non-resonance condition at infinity (5.25) yields $\varphi = 0$ showing that (u, v) is a zero mild solution of (5.28), as desired. This allows us to use Proposition 5.3.1 (i), which ends the proof of the assertion (i).

(ii) We shall show that, for any $\varepsilon \in (0, 1]$, the linearized equation (5.29) has no nonzero εT-periodic mild solutions. To this end, let us take an εT-periodic mild solution $(u, v) : [0, +\infty) \to \mathbb{X}$ of the equation (5.29). As before, we rewrite it as the evolution equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}^{(\alpha)}(u(t), v(t)), \quad t > 0,$$

where $\mathbb{A}^{(\alpha)} : D(\mathbb{A}^{(\alpha)}) \subset \mathbb{X} \to \mathbb{X}$ is given as follows

$$\mathbb{A}^{(\alpha)}(u, v) = (v, -(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha})u - \beta v) \quad \text{for} \quad (u, v) \in D(\mathbb{A}^{(\alpha)}) = D(\mathbb{A}),$$

and $\boldsymbol{\alpha} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the operator of multiplication by the function α. By the assumption, $V - \alpha$ is a Kato-Rellich type potential with a positive asymptotic bottom. Then we apply Lemma 4.2.3 and the non-resonance condition at zero (5.26) to get that (u, v) is a zero function, which means that, for any $\varepsilon \in (0, 1]$, the equation (5.29) does not admit nonzero εT-periodic mild solutions. Therefore, we may use Proposition 5.3.1 (ii), which completes the proof of the assertion (ii). □

5.4 Index formulae for autonomous equations

In this section, we consider equation (5.1) with a time-independent nonlinearity:

$$u_{tt} + \beta u_t = \Delta u - V(x)u + f(x, u), \quad t > 0, \ x \in \mathbb{R}^N. \tag{5.42}$$

Let $\Phi_t : \mathbb{X} \to \mathbb{X}$ denote the translation along trajectories operator associated with the corresponding evolution equation (5.2). Assume that f admits a linearization at infinity with the coefficient ω (see (5.20)). We denote the corresponding linearization by f^ω (see (5.21)), and write $\boldsymbol{\omega}$ for the multiplication operator by the function ω. Assume also that f admits a linearization at zero with the coefficient α (cf. (5.22)). We denote the linearization of f at zero by f^α (cf. (5.23)) and write $\boldsymbol{\alpha}$ for the multiplication operator by α.

We shall now prove the following index formulae for the equation (5.42).

Theorem 5.4.1. *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (f1) *and* (f2)*, and the function l from* (f1) *is such that* (5.7) *holds.*

(i) *Assume that f admits a linearization at infinity, and that its coefficient of linearization ω is a Kato-Rellich type function satisfying*

$$\varrho(V_\infty - \omega_\infty) > 0. \tag{5.43}$$

If the corresponding multiplication operator satisfies the non-resonance condition at infinity,

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}) = \{0\}, \tag{5.44}$$

then there exist constants $R_0 > 0$ and $t_0 > 0$ such that

$$\mathbb{A}(\overline{u}, \overline{v}) + (0, F(\overline{u})) \neq 0 \quad \textit{for} \quad \textit{all} \quad \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \geq R_0 \tag{5.45}$$

and

$$\Phi_t(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \textit{for all} \quad t \in (0, t_0] \quad \textit{and} \quad \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \geq R_0.$$

Moreover, for all $t \in (0, t_0]$ and $R \geq R_0$,

$$\operatorname{Ind}_C(\Phi_t, B_{\mathbb{X}}(0, R)) = (-1)^{m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})} \tag{5.46}$$

where $m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})$ denotes the sum of the multiplicities of the negative eigenvalues of the operator $-\Delta + \boldsymbol{V} - \boldsymbol{\omega}$.

(ii) *Assume that f admits a linearization at zero, and that its coefficient of linearization α is a Kato-Rellich type function satisfying*

$$\varrho(V_\infty - \alpha_\infty) > 0. \tag{5.47}$$

If the corresponding multiplication operator is such that the non-resonance condition at zero is satisfied,

$$\mathrm{Ker}(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) = \{0\}, \tag{5.48}$$

then there exist constants $r_0 > 0$ and $t_0 > 0$ such that

$$\mathbb{A}(\overline{u},\overline{v}) + (0, F(\overline{u})) \neq 0 \quad \text{for} \quad \text{all} \quad 0 < \|(\overline{u},\overline{v})\|_{\mathbb{X}} \leq r_0 \tag{5.49}$$

and

$$\Phi_t(\overline{u},\overline{v}) \neq (\overline{u},\overline{v}) \quad \text{for all} \quad t \in (0, t_0] \quad \text{and} \quad 0 < \|(\overline{u},\overline{v})\|_{\mathbb{X}} \leq r_0.$$

Moreover, for all $t \in (0, t_0]$ and $0 < r \leq r_0$,

$$\mathrm{Ind}_C(\Phi_t, B_{\mathbb{X}}(0,r)) = (-1)^{m_-(-\Delta+\boldsymbol{V}-\boldsymbol{\alpha})}. \tag{5.50}$$

Remark 5.4.2. Assume that function f and its coefficients of the linearizations, denoted by ω and α, are as in Theorem 5.4.1. Since ω and α are Kato-Rellich type functions (see (8a) and (8b)), it follows from Corollary 3.1.1 that $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are well-defined and bounded as the linear operators from $H^1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. Moreover, $V - \omega$ and $V - \alpha$ are also Kato-Rellich type potentials (see (5a) and (5b)). Thus, by Proposition 2.1.10 (ii), $-\Delta + \boldsymbol{V} - \boldsymbol{\omega}$ and $-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}$ are well-defined and self-adjoint operators in $L^2(\mathbb{R}^N)$. In view of the assumption that $\varrho(V_\infty - \omega_\infty) > 0$ and $\varrho(V_\infty - \alpha_\infty) > 0$, Proposition 2.1.12 (iii) shows that $\sigma(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}) \cap (-\infty, 0)$ and $\sigma(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) \cap (-\infty, 0)$ consist of the finite number of eigenvalues with finite multiplicities, and consequently the numbers $m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})$ and $m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha})$ are well-defined and finite. □

Proof of Theorem 5.4.1. (i) Let $\Psi_t : \mathbb{X} \times [0,1] \to \mathbb{X}$ be the translation along trajectories operator associated with the equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, (1-\mu)F(u(t)) + \mu\, \boldsymbol{\omega}\, u(t)\big), \quad t > 0. \tag{5.51}$$

Let us define $h : \mathbb{R}^N \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$h(x,u,\mu) = (1-\mu) f(x,u) + \mu\omega(x)u.$$

Since f is a Carathéodory function and ω is measurable, h is also a Carathéodory function (see condition $(\widetilde{C})$ on p. 53). Because f satisfies (f1) with a function l and $|\omega(x)| \leq l(x)$ for almost every $x \in \mathbb{R}^N$ (cf. Remark 5.3.2 (ii)), we see that h satisfies (h1) (see p. 53) with the same function l. Further, h satisfies (h2) (see p. 53) with the same function m_0 as f. Consequently, in view of Corollary 3.1.5, $H : H^1(\mathbb{R}^N) \times [0,1] \to L^2(\mathbb{R}^N)$ – the Nemytskii operator associated with function h – is well-defined, continuous and fulfills conditions (**H1**) and (**H2**) (see p. 54). Moreover, for $u \in H^1(\mathbb{R}^N)$ and $\mu \in [0,1]$,

$$H(u,\mu) = (1-\mu)F(u) + \mu\, \boldsymbol{\omega}\, u.$$

Therefore, by virtue of Theorem 3.4.1 (points (i) and (ii)), for any $t > 0$, the translation operator $\Psi_t : \mathbb{X} \times [0,1] \to \mathbb{X}$ associated with the equation (5.51) is well-defined and continuous. Observe that, by Corollary 3.3.4,

$$\chi_{L^2}(H(U \times [0,1])) \leq \widehat{\varrho}(l_\infty)\chi_{L^2}(U)$$

for any bounded $U \subset H^1(\mathbb{R}^N)$. Hence, Theorem 3.4.1 (iii) provides that, for any $t > 0$ and bounded subset $Z \subset \mathbb{X}$,

$$\chi_{s,V}\big(\Psi_t(Z \times [0,1])\big) \leq e^{-(\rho-\widehat{\varrho}(l_\infty)/\sqrt{d})t}\chi_{s,V}(Z). \tag{5.52}$$

We shall show that there exist $R_0 > 0$ and $t_0 > 0$ such that

$$\Psi_t(\overline{u},\overline{v},\mu) \neq (\overline{u},\overline{v}) \quad \text{for all } t \in (0,t_0],\ \mu \in [0,1],\ \text{and } \|(\overline{u},\overline{v})\|_{\mathbb{X}} \geq R_0. \tag{5.53}$$

Assume, by contradiction, that there are sequences $(t_n)_{n\geq 1}$ in $(0,+\infty)$, $(\mu_n)_{n\geq 1}$ in $[0,1]$ and $\big((\overline{u}_n,\overline{v}_n)\big)_{n\geq 1}$ in $\mathbb{X}$ such that $t_n \to 0$, $\rho_n = \|(\overline{u}_n,\overline{v}_n)\|_{\mathbb{X}} \to +\infty$ and $\Psi_{t_n}(\overline{u}_n,\overline{v}_n,\mu_n) = (\overline{u}_n,\overline{v}_n)$ for any $n \geq 1$. Therefore, for each $n \geq 1$, there exists the t_n-periodic mild solution $(u_n,v_n) : [0,+\infty) \to \mathbb{X}$ of the equation (5.51) with $\mu = \mu_n$ and $(u_n(0),v_n(0)) = (\overline{u}_n,\overline{v}_n)$. Let $(\widetilde{u}_n,\widetilde{v}_n) : [0,+\infty) \to \mathbb{X}$, $n \geq 1$, be defined by $(\widetilde{u}_n(t),\widetilde{v}_n(t)) = \rho_n^{-1}(u_n(t),v_n(t))$. By the Duhamel formula, we easily see that $(\widetilde{u}_n,\widetilde{v}_n)$, $n \geq 1$, are the t_n-periodic mild solutions of the Cauchy problem

$$\begin{cases} (\dot{u}(t),\dot{v}(t)) = \mathbb{A}(u(t),v(t)) + \Big(0,(1-\mu_n)\rho_n^{-1}F(\rho_n u(t)) + \mu_n\,\boldsymbol{\omega}\,u(t)\Big), & t > 0, \\ (u(0),v(0)) = \rho_n^{-1}(\overline{u}_n,\overline{v}_n). \end{cases}$$

Without loss of generality, we may assume that $\mu_n \to \mu_0$ for some $\mu_0 \in [0,1]$. Let us define $f_n^\omega : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, by the formula

$$f_n^\omega(x,u) = (1-\mu_n)\rho_n^{-1}f(x,\rho_n u) + \mu_n\omega(x)u,$$

and let $f^\omega : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be given by $f^\omega(x,u) = \omega(x)u$. By virtue of Lemma 5.3.3 (i), f_n^ω, $n \geq 1$, are Carathéodory functions satisfying condition (f1) with the same mapping l as f, and condition (f2) with the function Cm_0, where m_0 is the function appearing in (f2) for f, and $C > 0$ is such that $\rho_n^{-1} \leq C$ for all $n \geq 1$. Moreover, f^ω is a Carathéodory function satisfying (f1) with the same l as f, and (f2) with $m_0 \equiv 0$. Finally, $f_n^\omega \to f^\omega$ in the sense of condition (f0).

Given $n \geq 1$, we denote by $F_n^\omega : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ the Nemytskii operators associated with the functions f_n^ω, and by $F^\omega : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ we denote the Nemytskii operator associated with the function f^ω. Observe that, for $n \geq 1$ and $u \in H^1(\mathbb{R}^N)$,

$$(1-\mu_n)\rho_n^{-1}F(\rho_n u) + \mu_n\,\boldsymbol{\omega}\,u = F_n^\omega(u) \quad \text{and} \quad \boldsymbol{\omega}\,u = F^\omega(u).$$

We apply Proposition 5.1.2 (ii) for the sequence of functions f_n^ω, $n \geq 1$, and the function f^ω, obtaining (passing to a subsequence if necessary) that $\rho_n^{-1}(\overline{u}_n,\overline{v}_n) \xrightarrow{n\to\infty} (\varphi,\xi) \in D(\mathbb{A})$ such that

$$\mathbb{A}(\varphi,\xi) + \big(0,\boldsymbol{\omega}\,\varphi\big) = 0.$$

Hence, $\xi = 0$, $\varphi \in H^2(\mathbb{R}^N)$ and

$$(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})\varphi = 0.$$

Moreover, since, for all $n \geq 1$, $\|\rho_n^{-1}(\overline{u}_n,\overline{v}_n)\|_{\mathbb{X}} = 1$, one has $\|(\varphi,\xi)\|_{\mathbb{X}} = 1$ and consequently $\varphi \neq 0$. Thus, we have reached a contradiction with the non-resonance condition at infinity (5.44), which proves (5.53).

Suppose now, to the contrary, that (5.45) is not satisfied, i.e., for any $R > 0$ there exists $(\overline{u},\overline{v}) \in D(\mathbb{A})$ such that $\|(\overline{u},\overline{v})\|_{\mathbb{X}} \geq R$ and

$$\mathbb{A}(\overline{u},\overline{v}) + (0,F(\overline{u})) = 0. \tag{5.54}$$

Let $R_0 > 0$ and $t_0 > 0$ be such that (5.53) holds. Therefore, there exists $(\overline{u}_0,\overline{v}_0) \in D(\mathbb{A})$ satisfying $\|(\overline{u}_0,\overline{v}_0)\|_{\mathbb{X}} \geq R_0$ and (5.54). Then the constant map $(u_0,v_0) : [0,+\infty) \to \mathbb{X}$ defined by $(u_0(t),v_0(t)) = (\overline{u}_0,\overline{v}_0)$ is the mild solution of (5.51) with $\mu = 0$ and $(u(0),v(0)) = (\overline{u}_0,\overline{v}_0)$ – see Lemma 1.2.3. Hence, $\Psi_{t_0}(\overline{u}_0,\overline{v}_0,0) = (\overline{u}_0,\overline{v}_0)$, which contradicts (5.53) and proves (5.45).

It follows from the assumed property (5.7), the inequality (5.52), and the property (5.53) that there are $R_0 > 0$ and $t_0 > 0$ such that, for any $t \in (0, t_0]$ and $R \geq R_0$, the map Ψ_t, restricted to $\overline{B_{\mathbb{X}}(0,R)} \times [0,1]$, is an admissible homotopy for the index for k-set contractions. Therefore, the homotopy invariance (see Theorem A.4.8) yields

$$\mathrm{Ind}_C\big(\Phi_t, B_{\mathbb{X}}(0,R)\big) = \mathrm{Ind}_C\big(\Psi_t(\cdot,0), B_{\mathbb{X}}(0,R)\big) = \mathrm{Ind}_C\big(\Psi_t(\cdot,1), B_{\mathbb{X}}(0,R)\big). \tag{5.55}$$

Observe that $\Psi_t(\cdot, 1)$ is the translation operator associated with the linear equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + (0, \boldsymbol{\omega}\, u(t)), \quad t > 0.$$

Moreover, the linear operator $\mathbb{L}_{\boldsymbol{\omega}} : \mathbb{X} \to \mathbb{X}$ given by the formula $\mathbb{L}_{\boldsymbol{\omega}}(u,v) = (0, \boldsymbol{\omega}\, u)$ is bounded, because $\boldsymbol{\omega} : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a bounded linear operator – see Remark 5.3.2 (iv). Therefore, by Theorem 1.1.8, the perturbed linear operator $\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}} : D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}}) \subset \mathbb{X} \to \mathbb{X}$, defined as

$$(\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}})(u,v) = \mathbb{A}(u,v) + \mathbb{L}_{\boldsymbol{\omega}}(u,v) \quad \text{for} \quad (u,v) \in D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}}) = D(\mathbb{A}),$$

is also the generator of a C_0-semigroup $\{e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\omega}})}\}_{t \geq 0}$, and, for any $t > 0$, $\Psi_t(\cdot,1) = e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\omega}})}$. Next, note that, for all $(u,v) \in D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}})$,

$$(\mathbb{A} + \mathbb{L}_{\boldsymbol{\omega}})(u,v) = \Big(v, -\big(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}\big)u - \beta v\Big)$$

and, by the assumption, $V - \omega$ is a Kato-Rellich type potential such that $\varrho(V_\infty - \omega_\infty) > 0$. Then, in view of Corollary 2.4.5, we get that, for any $t > 0$, the map $\Psi_t(\cdot,1)$, restricted to $\overline{B_{\mathbb{X}}(0,R)} \times [0,1]$, is admissible for the index for k-set contractions, and, for all $t > 0$ and $R > 0$,

$$\mathrm{Ind}_C\big(\Psi_t(\cdot,1), B_{\mathbb{X}}(0,R)\big) = \mathrm{Ind}_C\big(e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\omega}})}, B_{\mathbb{X}}(0,R)\big) = (-1)^{m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\omega})}.$$

Combining this with equation (5.55), we obtain the desired index formula (5.46).

(ii) Let $\Psi_t : \mathbb{X} \times [0,1] \to \mathbb{X}$ be the translation operator associated with the equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \big(0, (1-\mu)F(u(t)) + \mu\, \boldsymbol{\alpha}\, u(t)\big), \quad t > 0. \tag{5.56}$$

We define $h : \mathbb{R}^N \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by the formula

$$h(x,u,\mu) = (1-\mu) f(x,u) + \mu \alpha(x) u.$$

Note that, as in the point (i), h is a Carathéodory function satisfying (h1) with the same l as f. By Remark 5.3.2 (i), $f(x,0) = 0$ for almost every $x \in \mathbb{R}^N$, hence h satisfies (h2) with $m_0 \equiv 0$. Thus, based on Theorem 3.4.1, for any $t > 0$, Ψ_t is a well-defined continuous k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$.

We shall show that there are $r_0 > 0$ and $t_0 > 0$ such that

$$\Psi_t(\overline{u}, \overline{v}, \mu) \neq (\overline{u}, \overline{v}) \ \text{ for all } t \in (0, t_0],\ \mu \in [0,1],\ \text{and } 0 < \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \leq r_0. \tag{5.57}$$

Assume, by contradiction, that (5.57) is not satisfied. Therefore, we obtain sequences $(t_n)_{n\geq 1}$ in $(0,+\infty)$, $(\mu_n)_{n \geq 1}$ in $[0,1]$ and $\big((\overline{u}_n, \overline{v}_n)\big)_{n \geq 1}$ in $\mathbb{X}$ such that $t_n \to 0$, $\rho_n = \|(\overline{u}_n, \overline{v}_n)\|_{\mathbb{X}} \to 0$, and $\Psi_{t_n}(\overline{u}_n, \overline{v}_n, \mu_n) = (\overline{u}_n, \overline{v}_n)$ for any $n \geq 1$. Consequently, given $n \geq 1$, there exists the t_n-periodic mild solution $(u_n, v_n) : [0,+\infty) \to \mathbb{X}$ of the equation (5.56) with $\mu = \mu_n$ and $(u_n(0), v_n(0)) = (\overline{u}_n, \overline{v}_n)$. Let $(\widetilde{u}_n, \widetilde{v}_n) : [0,+\infty) \to \mathbb{X}$, $n \geq 1$, be defined as $(\widetilde{u}_n(t), \widetilde{v}_n(t)) = \rho_n^{-1}(u_n(t), v_n(t))$. Then, for any $n \geq 1$, $(\widetilde{u}_n, \widetilde{v}_n)$ is the t_n-periodic mild solution of the Cauchy problem

$$\begin{cases} (\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \Big(0, (1-\mu_n)\rho_n^{-1} F(\rho_n u(t)) + \mu_n\, \boldsymbol{\alpha}\, u(t)\Big), & t > 0, \\ (u(0), v(0)) = \rho_n^{-1}(\overline{u}_n, \overline{v}_n). \end{cases}$$

Without loss of generality we may assume that $\mu_n \to \mu_0$ for some $\mu_0 \in [0,1]$. Let us define functions $f_n^\alpha : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, by the formula

$$f_n^\alpha(x,u) = (1-\mu_n)\rho_n^{-1} f(x, \rho_n u) + \mu_n \alpha(x) u,$$

and let $f^\alpha : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be given by $f^\alpha(x,u) = \alpha(x)u$. It follows from Lemma 5.3.3 (ii) that, for any $n \geq 1$, f_n^α is a Carathéodory function satisfying (f1) with the same mapping l as f, and (f2) with the function $m_0 \equiv 0$. By Remark 5.3.2 (iii), f^α is also a Carathéodory function satisfying (f1) with the same function l as f, and (f2) with the function $m_0 \equiv 0$. Furthermore, Lemma 5.3.3 (ii) shows that $f_n^\alpha \to f^\alpha$ in the sense of condition (f0). For each $n \geq 1$, we define the map $F_n^\alpha : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ as the Nemytskii operator associated with the function f_n^α. Moreover, we define $F^\alpha : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ as the Nemytskii operator associated with f^α. Note that, for all $n \geq 1$ and $u \in H^1(\mathbb{R}^N)$,

$$F_n(u) = (1-\mu_n)\rho_n^{-1} F(\rho_n u) + \mu_n \boldsymbol{\alpha}\, u \quad \text{and} \quad F^\alpha(u) = \boldsymbol{\alpha}\, u.$$

Therefore, it follows from Proposition 5.1.2 (ii) that, passing to a subsequence if necessary, $\rho_n^{-1}(\overline{u}_n, \overline{v}_n) \xrightarrow{n\to\infty} (\varphi, \xi) \in D(\mathbb{A})$ such that

$$\mathbb{A}(\varphi,\xi) + (0, F^\alpha(\varphi)) = 0.$$

Observe that the equation above is equivalent to

$$\xi = 0 \quad \text{and} \quad (\Delta - \boldsymbol{V} + \boldsymbol{\alpha})\varphi = 0.$$

As in the proof of the point (i), we see that $\|(\varphi,\xi)\|_{\mathbb{X}} = 1$, hence $\varphi \neq 0$. We thus arrive at a contradiction with the non-resonance condition at zero (5.48), which proves property (5.57).

Now, suppose to the contrary that (5.49) does not hold. Let $r_0 > 0$ and $t_0 > 0$ be such that (5.57) is satisfied. Then there exists $(\overline{u}_0, \overline{v}_0) \in D(\mathbb{A})$ such that $0 < \|(\overline{u}_0, \overline{v}_0)\|_{\mathbb{X}} \leq r_0$ and $\mathbb{A}(\overline{u}_0, \overline{v}_0) + (0, F(\overline{u}_0)) = 0$. By Lemma 1.2.3, the constant function $(u_0, v_0) : [0, +\infty) \to \mathbb{X}$ given by $(u_0(t), v_0(t)) = (\overline{u}_0, \overline{v}_0)$, is the mild solution of the equation (5.56) with $\mu = 0$. This yields $\Psi_{t_0}(\overline{u}_0, \overline{v}_0, 0) = (\overline{u}_0, \overline{v}_0)$, hence we reached a contradiction with (5.57), which proves (5.49).

Since, for all $t \in (0, t_0]$ and $r \in (0, r_0]$, Ψ_t is a continuous k-set contraction with respect to the Hausdorff measure of non-compactness $\chi_{s,V}$, and Ψ_t satisfies (5.57), we see that, for all $t \in (0, t_0]$ and $r \in (0, r_0]$, the map Ψ_t, restricted to $\overline{B_{\mathbb{X}}(0,r)} \times [0,1]$, is an admissible homotopy for the index for k-set contractions. Therefore, the homotopy invariance yields (see Theorem A.4.8), for all $t \in (0, t_0]$ and $r \in (0, r_0]$,

$$\mathrm{Ind}_C(\Phi_t, B_{\mathbb{X}}(0,r)) = \mathrm{Ind}_C(\Psi_t(\cdot, 0), B_{\mathbb{X}}(0,r)) = \mathrm{Ind}_C(\Psi_t(\cdot,1), B_{\mathbb{X}}(0,r)). \tag{5.58}$$

Observe that $\Psi_t(\cdot, 1)$ is the translation operator associated with

$$(\dot{u}(t), \dot{v}(t)) = \mathbb{A}(u(t), v(t)) + \mathbb{L}_{\boldsymbol{\alpha}}(u(t), v(t)), \quad t > 0$$

where the bounded linear operator $\mathbb{L}_{\boldsymbol{\alpha}} : \mathbb{X} \to \mathbb{X}$ is given by $\mathbb{L}_{\boldsymbol{\alpha}}(u,v) = (0, \boldsymbol{\alpha}\, u)$. In view of Theorem 1.1.8, the linear operator $\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}} : D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}}) \subset \mathbb{X} \to \mathbb{X}$, defined as

$$(\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}})(u,v) = \mathbb{A}(u,v) + \mathbb{L}_{\boldsymbol{\alpha}}(u,v) \quad \text{for} \quad (u,v) \in D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}}) = D(\mathbb{A}),$$

is the generator of a C_0-semigroup $\{e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\alpha}})}\}_{t\geq 0}$, and, for all $t > 0$, $e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\alpha}})} = \Psi_t(\cdot, 1)$. Moreover, we note that, for all $(u,v) \in D(\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}})$,

$$(\mathbb{A} + \mathbb{L}_{\boldsymbol{\alpha}})(u,v) = \Big(v, -\big(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}\big)u - \beta v\Big)$$

and, by the assumption, $V - \alpha$ is a Kato-Rellich type potential such that $\varrho(V_\infty - \alpha_\infty) > 0$. Hence, Corollary 2.4.5 yields, for all $t > 0$ and $r > 0$,

$$\mathrm{Ind}_C(\Psi_t(\cdot,1), B_{\mathbb{X}}(0,r)) = \mathrm{Ind}_C(e^{t(\mathbb{A}+\mathbb{L}_{\boldsymbol{\alpha}})}, B_{\mathbb{X}}(0,r)) = (-1)^{m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha})}.$$

Combining this with (5.58), we obtain (5.50), the index formula for the case in which f admits a linearization at zero. This completes the proof of the theorem. □

We conclude this section with the following remark regarding the validity of the assumptions (5.43) and (5.47).

Remark 5.4.3.

(i) Let $l : \mathbb{R}^N \to \mathbb{R}$ be a Kato-Rellich type function and let $k : \mathbb{R}^N \to \mathbb{R}$ be a measurable function such that $|k(x)| \le l(x)$ for almost every $x \in \mathbb{R}^N$. Then k is also a Kato-Rellich type function, in particular, Kato-Rellich type potential. Indeed, we define k_∞, $k_0 : \mathbb{R}^N \to \mathbb{R}$ as follows

$$k_\infty(x) = k(x)\,\mathbf{1}_{E_\infty}(x) \quad \text{and} \quad k_0(x) = k(x)\,\mathbf{1}_{E_0 \setminus E_\infty}(x) \tag{5.59}$$

where

$$E_\infty = \{x \in \mathbb{R}^N : |k(x)| \le 2l_\infty(x)\} \quad \text{and} \quad E_0 = \{x \in \mathbb{R}^N : |k(x)| \le 2l_0(x)\}.$$

Obviously, $k_\infty \in L^\infty(\mathbb{R}^N)$ and $k_0 \in L^p(\mathbb{R}^N)$ with $p \ge 2$ as in (8b). Further, we claim that $k(x) = k_\infty(x) + k_0(x)$ for almost every $x \in \mathbb{R}^N$. To this end, observe that

$$\mathbb{R}^N \setminus (E_\infty \cup E_0) \subset \{x \in \mathbb{R}^N : |k(x)| > l(x)\}.$$

Therefore, the set $\mathbb{R}^N \setminus (E_\infty \cup E_0)$ has measure zero. On the other hand, we have, for any $x \in E_\infty \cup E_0$,

$$k(x) = k(x)(\mathbf{1}_{E_\infty}(x) + \mathbf{1}_{E_0 \setminus E_\infty}(x)) = k_\infty(x) + k_0(x).$$

(ii) Assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type potential and $l : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type function such that $2\widehat{\varrho}(l_\infty) < \varrho(V_\infty)$. If $k : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that $|k(x)| \le l(x)$ for almost every $x \in \mathbb{R}^N$, then k is also a Kato-Rellich type function satisfying

$$\varrho(V_\infty - k_\infty) = \lim_{R \to +\infty} \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} (V_\infty - k_\infty) > 0$$

where $k_\infty : \mathbb{R}^N \to \mathbb{R}$ is given by (5.59).

In fact, from the point (i) we get that k is Kato-Rellich type function with the decomposition $k = k_\infty + k_0$ given by (5.59). Observe that, we have, for any $R > 0$,

$$\begin{aligned} \varrho(V_\infty - k_\infty) &\ge \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} (V_\infty - k_\infty) \\ &\ge \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty + \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} (-k_\infty) = \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R)} V_\infty - \operatorname*{ess\,sup}_{\mathbb{R}^N \setminus D(0,R)} k_\infty. \end{aligned} \tag{5.60}$$

Given $\varepsilon > 0$, we take $R_\varepsilon > 0$ such that

$$\operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R_\varepsilon)} V_\infty > \varrho(V_\infty) - \varepsilon \tag{5.61}$$

and

$$\operatorname*{ess\,sup}_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} l_\infty < \widehat{\varrho}(l_\infty) + \varepsilon.$$

Note that $k_\infty(x) \le 2l_\infty(x)$ for almost every $x \in \mathbb{R}^N$, hence, we have

$$\operatorname*{ess\,sup}_{\mathbb{R}^N \setminus D(0,R_\varepsilon)} k_\infty \le 2 \operatorname*{ess\,sup}_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} l_\infty < 2\widehat{\varrho}(l_\infty) + 2\varepsilon. \tag{5.62}$$

We apply (5.61) and (5.62) to (5.60), which gives

$$\begin{aligned} \varrho(V_\infty - k_\infty) &\ge \operatorname*{ess\,inf}_{\mathbb{R}^N \setminus D(0,R_\varepsilon)} V_\infty - \operatorname*{ess\,sup}_{\mathbb{R}^N \setminus D(0,R_\varepsilon)} k_\infty > \varrho(V_\infty) - \varepsilon - \operatorname*{ess\,sup}_{\mathbb{R}^N \setminus D(0,R_\varepsilon)} k_\infty \\ &> \varrho(V_\infty) - \varepsilon - 2\widehat{\varrho}(l_\infty) - 2\varepsilon = \varrho(V_\infty) - 2\widehat{\varrho}(l_\infty) - 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and, by the assumption, $2\widehat{\varrho}(l_\infty) < \varrho(V_\infty)$, we obtain

$$\varrho(V_\infty - k_\infty) \ge \varrho(V_\infty) - 2\widehat{\varrho}(l_\infty) > 0,$$

as claimed.

(iii) If we assume additionally in Theorem 5.4.1 that $\varrho(V_\infty) > 2\widehat{\varrho}(l_\infty)$, then in the point (i) we may drop the assumption that the coefficient of linearization ω is a Kato-Rellich type function satisfying (5.43), and in the point (ii) we may drop the assumption that the coefficient of linearization α is a Kato-Rellich type function satisfying (5.47).

To this end, we note, by Remark 5.4.2 (ii), that the coefficients ω and α satisfy $|\omega(x)| \le l(x)$ and $|\alpha(x)| \le l(x)$, for almost every $x \in \mathbb{R}^N$. Therefore, by virtue of the point (i), ω and α are Kato-Rellich type functions with the decompositions $\omega = \omega_\infty + \omega_0$ and $\alpha = \alpha_\infty + \alpha_0$ given by (5.59). Consequently, $V - \omega$ and $V - \alpha$ are Kato-Rellich type potentials. Since $2\widehat{\varrho}(l_\infty) < \varrho(V_\infty)$, from the point (ii) we deduce (5.43) and (5.47), as desired.

(iv) Consider the function $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by

$$V(x) = -\frac{a}{|x|} + b \tag{5.63}$$

where $a, b > 0$, and the map $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, u) = g(u) + h(u|x|^{-\gamma})$$

where $\gamma \in (0, 1)$, and the functions $g,\ h : \mathbb{R} \to \mathbb{R}$ are as defined in Remark 5.6.3. Then, in view of Example 2.1.1, the function V is a Kato-Rellich type potential, which admits a decomposition $V = V_\infty + V_0$ such that $\varrho(V_\infty) = b$. One can show (see the part of the proof of Theorem 5.6.1 on p. 122, and Remark 5.6.3) that f is a Carathéodory function satisfying condition (f1) with the function $l : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by $l(x) = 1 + |x|^{-\gamma}$, and condition (f2) with $m_0 \equiv 0$. Moreover, l is a Kato-Rellich type function, which admits a decomposition $l = l_\infty + l_0$ such that $\widehat{\varrho}(l_\infty) = 1$. Hence, if the number $b > 0$ from (5.63) satisfies $b > 2$, then $\varrho(V_\infty) > 2\widehat{\varrho}(l_\infty)$, i.e., the condition in point (iii) is satisfied. □

5.5 Proof of Theorem II

In this section, we prove an extended version of Theorem II that covers the case where equation (5.1) is considered only under the non-resonance condition at infinity.

Theorem 5.5.1. *Assume that $V : \mathbb{R}^N \to \mathbb{R}$, where $N \ge 1$, is a Kato-Rellich type potential such that the asymptotic bottom of V_∞ is positive (see conditions* (5a), (5b) *and* (7)*). Suppose that a time T-periodic Carathéodory function $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (f1) *and* (f2)*, and that the function l from* (f1) *satisfies* (13) *(see conditions (P) on p. ix, (C) on p. viii,* (f1) *on p. ix and* (f2) *on p. ix).*

(i) Suppose that f admits a linearization at infinity with a coefficient denoted by $\omega : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $\widehat{\omega} : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type function satisfying

$$\varrho(V_\infty - \widehat{\omega}_\infty) > 0,$$

and that the non-resonance condition at infinity holds (see (5.25)*). If, for every $\varepsilon \in (0, 1]$, the linear damped wave equation* (5.28) *admits no nonzero εT-periodic mild solutions, then the nonlinear damped wave equation* (5.1) *admits a T-periodic mild solution.*

(ii) Suppose all assumptions from (i) hold, and, in addition, that f admits a linearization at zero with a coefficient denoted by $\alpha : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $\widehat{\alpha} : \mathbb{R}^N \to \mathbb{R}$ is a Kato-Rellich type function satisfying

$$\varrho(V_\infty - \widehat{\alpha}_\infty) > 0,$$

and the non-resonance condition at zero holds (see (5.26)*). If, for every* $\varepsilon \in (0,1]$*, the linear damped wave equation* (5.29) *admits no nonzero* εT*-periodic mild solutions, and if (cf.* (18)*)*

$$m_-(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\omega}}) \not\equiv m_-(-\Delta + \boldsymbol{V} - \widehat{\boldsymbol{\alpha}}) \pmod 2 \tag{5.64}$$

where $m_-(\cdot)$ *denotes the total multiplicity of negative eigenvalues, then the nonlinear damped wave equation* (5.1) *admits a nonzero* T*-periodic mild solution.*

The proof relies on the following lemma.

Lemma 5.5.2. *Assume that* f *is a* T*-periodic Carathéodory function satisfying conditions* (f1) *and* (f2)*, and let* $\widehat{f}$ *be given by* (5.5)*. If* f *admits a linearization at infinity with the coefficient denoted by* ω*, then* $\widehat{f}$ *admits a linearization at infinity with the coefficient* $\widehat{\omega}$ *given by* (5.24)*. Similarly, if* f *admits a linearization at zero with the coefficient* α*, then* $\widehat{f}$ *admits a linearization at zero with the coefficient* $\widehat{\alpha}$ *given by* (5.24)*.*

Proof. Let us take a sequence $(u_n)_{n\geq 1}$ in $\mathbb{R}\setminus\{0\}$ such that $|u_n| \to +\infty$. By the conditions (f1) and (f2), we obtain

$$\left|\frac{f(t,x,u_n)}{u_n}\right| \leq \frac{l(x)|u_n| + m_0(x)}{|u_n|} \leq l(x) + Cm_0(x) \quad \text{for almost every } x \in \mathbb{R}^N$$

where constant $C > 0$ is such that $|u_n|^{-1} \leq C$ for all $n \geq 1$. Consequently, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \frac{\widehat{f}(x,u_n)}{u_n} = \lim_{n\to\infty} \frac{1}{T}\int_0^T \frac{f(t,x,u_n)}{u_n}\,dt = \frac{1}{T}\int_0^T \omega(t,x)\,dt = \widehat{\omega}(x) \quad \text{for almost every } x \in \mathbb{R}^N$$

i.e., $\widehat{f}$ admits a linearization at infinity and $\widehat{\omega}$ is its coefficient of linearization.

Next, let us take a sequence $(u_n)_{n\geq 1}$ in $\mathbb{R}\setminus\{0\}$ such that $u_n \to 0$. By the condition (f1) we have

$$\left|\frac{f(t,x,u_n)}{u_n}\right| \leq \frac{l(x)|u_n| + |f(t,x,0)|}{|u_n|} = l(x) \quad \text{for almost every } x \in \mathbb{R}^N$$

because, in view of Remark 5.3.2 (i), $f(t,x,0) = 0$, for almost every $x \in \mathbb{R}^N$ and all $t \geq 0$. This allows us to apply the Dominated Convergence Theorem obtaining that $\widehat{f}$ admits a linearization at zero and $\widehat{\alpha}$ is its coefficient of linearization. Therefore, the proof of the lemma is completed. □

Proof of Theorem 5.5.1. (i) Based on Proposition 3.1.2 (ii), $\widehat{f}$ is a Carathéodory function satisfying the conditions (f1) and (f2) with the same functions as f. By the assumption, $\widehat{\omega}$ is a Kato-Rellich type function such that $\varrho(V_\infty - \widehat{\omega}_\infty) > 0$ and the non-resonance condition at infinity (5.25) is satisfied. Moreover, in view of Lemma 5.5.2, $\widehat{f}$ admits a linearization at infinity, and $\widehat{\omega}$ is its coefficient of linearization. This allows us to apply Theorem 5.4.1 (i) for equation (5.42) with the nonlinearity $\widehat{f}$. Hence, there exist $R_0 > 0$ and $t_0 > 0$ such that

$$\mathbb{A}(u,v) + (0, \widehat{F}(u)) \neq 0 \quad \text{for} \quad (u,v) \in D(\mathbb{A}) \cap \partial Z \tag{5.65}$$

where $Z = B_{\mathbb{X}}(0,R_0)$, and

$$\widehat{\Phi}_t(\overline{u},\overline{v}) \neq (\overline{u},\overline{v}) \quad \text{for all} \quad t \in (0,t_0] \quad \text{and} \quad \|(\overline{u},\overline{v})\|_{\mathbb{X}} \geq R_0.$$

Moreover, we have, for all $t \in (0,t_0]$,

$$\mathrm{Ind}_C(\widehat{\Phi}_t, B_{\mathbb{X}}(0,R_0)) = (-1)^{m_-(-\Delta+\boldsymbol{V}-\widehat{\boldsymbol{\omega}})}.$$

By Proposition 5.3.1 (i), increasing $R_0 > 0$ if necessary, for any $\varepsilon \in (0, 1]$, the frequency changing evolution equation (5.3) does not admit εT-periodic mild solutions $(u, v) : [0, +\infty) \to \mathbb{X}$ with $\|(u(0), v(0))\|_{\mathbb{X}} \geq R_0$. In particular, one has

$$\Phi_T(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \text{for} \quad \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \geq R_0. \tag{5.66}$$

This enables us to apply Theorem 5.2.2 with the set $Z = B_{\mathbb{X}}(0, R_0)$. Hence, decreasing $t_0 > 0$ if necessary, one has, for all $t \in (0, t_0]$,

$$\mathrm{Ind}_C(\Phi_T, B_{\mathbb{X}}(0, R_0)) = \mathrm{Ind}_C(\widehat{\Phi}_t, B_{\mathbb{X}}(0, R_0)) = (-1)^{m_-(-\boldsymbol{\Delta}+\boldsymbol{V}-\widehat{\boldsymbol{\omega}})}, \tag{5.67}$$

which, by the existence property of the fixed-point index (cf. Theorem A.4.8), implies that equation (5.1) admits a T-periodic mild solution.

(ii) By the assumption, $\widehat{\alpha}$ is a Kato-Rellich type function such that $\varrho(V_\infty - \widehat{\alpha}_\infty) > 0$ and the non-resonance condition at zero (5.26) is satisfied. Moreover, in view of Lemma 5.5.2, $\widehat{f}$ admits a linearization at zero and $\widehat{\alpha}$ is its coefficient of linearization. Thus, by Theorem 5.4.1 (ii), there exist $r_0 > 0$ and $t_0 > 0$ such that the condition (5.65) is satisfied with $Z = B_{\mathbb{X}}(0, r_0)$, and

$$\widehat{\Phi}_t(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \text{for all} \quad t \in (0, t_0] \quad \text{and} \quad 0 < \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \leq r_0.$$

Moreover, we have, for all $t \in (0, t_0]$,

$$\mathrm{Ind}_C(\widehat{\Phi}_t, B_{\mathbb{X}}(0, r_0)) = (-1)^{m_-(-\boldsymbol{\Delta}+\boldsymbol{V}-\widehat{\boldsymbol{\alpha}})}.$$

In view of Proposition 5.3.1 (ii), decreasing $r_0 > 0$ if necessary, for any $\varepsilon \in (0, 1]$, the frequency changing equation (5.3) does not admit εT-periodic mild solutions $(u, v) : [0, +\infty) \to \mathbb{X}$ with $0 < \|(u(0), v(0))\|_{\mathbb{X}} \leq r_0$. In particular, we have

$$\Phi_T(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v}) \quad \text{for} \quad 0 < \|(\overline{u}, \overline{v})\|_{\mathbb{X}} \leq r_0. \tag{5.68}$$

Applying Theorem 5.2.2 with the set $Z = B_{\mathbb{X}}(0, r_0)$, we can decrease $t_0 > 0$ if necessary to obtain, for all $t \in (0, t_0]$,

$$\mathrm{Ind}_C(\Phi_T, B_{\mathbb{X}}(0, r_0)) = \mathrm{Ind}_C(\widehat{\Phi}_t, B_{\mathbb{X}}(0, r_0)) = (-1)^{m_-(-\boldsymbol{\Delta}+\boldsymbol{V}-\widehat{\boldsymbol{\alpha}})}. \tag{5.69}$$

Without loss of generality we can decrease $r_0 > 0$ such that $r_0 < R_0$. From (5.66) and (5.68), it follows that $\Phi_T(\overline{u}, \overline{v}) \neq (\overline{u}, \overline{v})$ for $(\overline{u}, \overline{v}) \in \partial Z$, where

$$Z = B_{\mathbb{X}}(0, R_0) \setminus \overline{B_{\mathbb{X}}(0, r_0)}.$$

Then, in view of the additivity property of the index (see Theorem A.4.8), equations (5.67), (5.69), and the assumption (5.64), we finally obtain

$$\begin{aligned}\mathrm{Ind}_C(\Phi_T, Z) &= \mathrm{Ind}_C(\Phi_T, B_{\mathbb{X}}(0, R_0)) - \mathrm{Ind}_C(\Phi_T, B_{\mathbb{X}}(0, r_0)) \\ &= (-1)^{m_-(-\boldsymbol{\Delta}+\boldsymbol{V}-\widehat{\boldsymbol{\omega}})} - (-1)^{m_-(-\boldsymbol{\Delta}+\boldsymbol{V}-\widehat{\boldsymbol{\alpha}})} \neq 0.\end{aligned}$$

This shows that Φ_T has a fixed-point in Z, i.e., equation (5.1) admits a nonzero T-periodic mild solution. This completes the proof of Theorem 5.5.1. □

Combining results from Chapter 4 with those from Chapter 5 we obtain a criterion for the existence of a nonzero T-periodic mild solution of equation (1) assuming that equation (1) is at resonance at infinity and the non-resonance condition at zero is satisfied.

Remark 5.5.3. Assume that $N \geq 3$, where N is the dimension of $\mathbb{R}^N$, V is a Kato-Rellich type potential with a positive asymptotic bottom, and that the nonlinearity f is a T-periodic Carathéodory function satisfying conditions (f1) and (f2). Suppose that equation (1) is at resonance at infinity (see (4)) and the non-resonance condition at zero is satisfied, where the coefficient of linearization at zero, denoted by α, is time-independent. Moreover, assume that function l from condition (f1) is such that (13) holds.

(i) If the Landesman-Lazer type condition $(LL)'_+$ or the strong resonance type condition $(SR)'_+$ is satisfied, and

$$m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) \not\equiv m_-(-\Delta + \boldsymbol{V}) + \dim X_0 \pmod 2,$$

then equation (1) admits a nonzero T-periodic mild solution.

(ii) If the Landesman-Lazer type condition $(LL)'_-$ or the strong resonance type condition $(SR)'_-$ is satisfied, and

$$m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) \not\equiv m_-(-\Delta + \boldsymbol{V}) \pmod 2,$$

then equation (1) admits a nonzero T-periodic mild solution. □

5.6 Applications to PDEs

We are concerned with the equation

$$u_{tt} + \beta u_t = \Delta u + \frac{a}{|x|}u - bu + g(u) + h(q(t,x)u), \quad t > 0, \ x \in \mathbb{R}^3 \tag{5.70}$$

where $\beta > 0$ is a damping coefficient, $a > 0$ and $-\lambda_2 < b < -\lambda_1$ are constants, λ_1 and λ_2 are the first and the second eigenvalue of the Schrödinger operator

$$-\Delta - \frac{a}{|x|}. \tag{5.71}$$

Functions g, $h : \mathbb{R} \to \mathbb{R}$ are such that $g(0) = h(0) = 0$, $g'(0) \neq 0$ and $h'(0) = 0$, and $q : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is T-periodic in time. Recall that, in view of Example 2.1.15, one has

$$\sigma_{\text{disc}}\left(-\Delta - \frac{a}{|x|}\right) = \{\lambda_n\}_{n\geq 1}, \quad \lambda_n = -\frac{a^2}{4n^2} \quad \text{for all } n \geq 1.$$

Theorem 5.6.1. *Let*

$$\widetilde{b} = \max\left\{-\lambda_2 + (\beta/3)^2, -\lambda_1 - \frac{(\beta/9)(\lambda_2 - \lambda_1)}{\beta/18 + \sqrt{\lambda_2 - \lambda_1 + (\beta/18)^2}}\right\}$$

and assume that $0 < \beta < 3\sqrt{\lambda_2 - \lambda_1}$, *and*

$$\widetilde{b} < b < -\lambda_1. \tag{5.72}$$

Then the equation (5.70) *admits a non-trivial T-periodic mild solution if*

(a) function g is bounded, differentiable at 0, satisfies the Lipschitz condition, and it is such that $g(0) = 0$ and $g'(0) \neq 0$;

(b) function h is bounded, differentiable at 0, satisfies the Lipschitz condition, and it is such that $h(0) = 0$ and $h'(0) = 0$;

(c) $q : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is a T-periodic Carathéodory function satisfying

$$|q(t,x)| \leq k(x) \quad \textit{for almost every } x \in \mathbb{R}^3 \textit{ and all } t \geq 0$$

where k is a Kato-Rellich type function satisfying $k_\infty(x) \geq 0$ and $k_0(x) \geq 0$, for almost every $x \in \mathbb{R}^3$;

(d) the following inequalities are satisfied

$$-(\beta/9)\sqrt{b+\lambda_2} \le g'(0) < b+\lambda_1 \tag{5.73}$$

and

$$L_g + L_h\,\widehat{\varrho}(k_\infty) \le (\beta/9)\sqrt{b+\lambda_2}$$

where $L_g > 0$ and $L_h > 0$ are the Lipschitz constants of the functions g and h, respectively.

Remark 5.6.2. Since, by the assumption, $\beta < 3\sqrt{\lambda_2 - \lambda_1}$, we see that

$$-\lambda_2 + (\beta/3)^2 < -\lambda_1.$$

Hence, $\widetilde{b} < -\lambda_1$ and the assumption (5.72) makes sense. Observe that the number

$$b_1 = \frac{(\beta/9)(\lambda_2 - \lambda_1)}{\beta/18 + \sqrt{\lambda_2 - \lambda_1 + (\beta/18)^2}}$$

is a solution of the equation

$$-(\beta/9)\sqrt{b+\lambda_2} = b + \lambda_1$$

satisfying $b_1 < -\lambda_1$. As the function

$$(-\lambda_2, -\lambda_1) \ni b \mapsto b + \lambda_1 + (\beta/9)\sqrt{b+\lambda_2}$$

is strictly increasing, one has

$$b + \lambda_1 + (\beta/9)\sqrt{b+\lambda_2} > 0 \quad \text{for } b \in (b_1, -\lambda_1).$$

In particular, for any b satisfying (5.72), we have $b + \lambda_1 > -(\beta/9)\sqrt{b+\lambda_2}$, hence, the assumption (5.73) is reasonable. □

Remark 5.6.3. Note that the function $g(u) = \sin u$ satisfies assumption (a) of Theorem 5.6.1, and the function $h(u) = 1 + \sin(u - \pi/2)$ satisfies assumption (b) of Theorem 5.6.1. Moreover, we observe that $L_g = 1$ is a Lipschitz constant for g, and $L_h = 1$ is a Lipschitz constant for h. The function $q(t,x) = \sin((\cos t)|x|^{-\gamma})$, where $\gamma \in (0,1)$, satisfies assumption (c) of Theorem 5.6.1, with the function k defined by $k(x) = |x|^{-\gamma}$. In addition, k is a Kato-Rellich type function, which admits a decomposition $k = k_\infty + k_0$ such that $\widehat{\varrho}(k_\infty) = 0$ – compare with Example 2.1.1. □

Proof of Theorem 5.6.1. We define

$$V(x) = -\frac{a}{|x|} + b \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

By Example 2.1.1, V is a Kato-Rellich type potential such that $\varrho(V_\infty) = b$. From the inequality (5.72) we get $b > -\lambda_2 + (\beta/3)^2 > 0$, i.e., the asymptotic bottom of V_∞ is positive.

Now, we claim that the mapping $f : [0,+\infty) \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(t,x,u) = g(u) + h(q(t,x)u) \tag{5.74}$$

for almost every $x \in \mathbb{R}^3$, all $t \ge 0$, and all $u \in \mathbb{R}$, is a T-periodic Carathéodory function satisfying (f1) with a Kato-Rellich type function l such that $\widehat{\varrho}(l_\infty) = L_g + L_h\,\widehat{\varrho}(k_\infty)$ and (f2) with $m_0 \equiv 0$. Indeed, for almost every $x \in \mathbb{R}^3$, all $t \ge 0$, and all u_1, $u_2 \in \mathbb{R}$, one has

$$\begin{aligned}|f(t,x,u_1) - f(t,x,u_2)| &\le |g(u_1) - g(u_2)| + |h(q(t,x)u_1) - h(q(t,x)u_2)| \\ &\le L_g|u_1 - u_2| + L_h|q(t,x)|\,|u_1 - u_2| \le (L_g + L_h k(x))|u_1 - u_2|.\end{aligned}$$

Defining $l(x) = L_g + L_h k(x)$, we see that $l = l_\infty + l_0$ is a Kato-Rellich type function with $l_\infty(x) = L_g + L_h k_\infty(x)$ and $l_0(x) = k_0(x)$. Moreover, one has $\widehat{\varrho}(l_\infty) = L_g + L_h\,\widehat{\varrho}(k_\infty)$. Since $g(0) = h(0) = 0$, the function f satisfies (f2) with $m_0 \equiv 0$.

We claim that function f admits linearizations at infinity with the coefficient $\omega(t, x) = 0$, for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$, and at zero with the coefficient $\alpha(t, x) = g'(0)$, for almost every $x \in \mathbb{R}^3$ and all $t \geq 0$. Indeed, since g and h are assumed to be bounded, we get immediately that $\omega \equiv 0$. Next, if $(t, x) \in [0, +\infty) \times \mathbb{R}^3$ is such that $q(t, x) \neq 0$, there holds

$$\frac{f(t, x, u)}{u} = \frac{g(u)}{u} + q(t, x) \cdot \frac{h(q(t, x)u)}{q(t, x)u} \to g'(0) \quad \text{as } u \to 0,$$

because $g(0) = h(0) = 0$ and $h'(0) = 0$. Hence, $\alpha(t, x) = g'(0)$. On the other hand, if $q(t, x) = 0$, we see immediately that also $\alpha(t, x) = g'(0)$, as desired. Obviously, ω and α are time-independent Kato-Rellich type functions consisting only of the $L^\infty(\mathbb{R}^3)$-part. By the assumption, $g'(0) < b + \lambda_1$, which implies

$$\varrho(V_\infty - \alpha) = b - g'(0) > b - b - \lambda_1 = -\lambda_1 > 0.$$

In view of Example 2.1.15 one has

$$\sigma_{\text{disc}}(-\Delta + \boldsymbol{V}) = \{\lambda_n + b\}_{n \geq 1} \quad \text{and} \quad \sigma_{\text{ess}}(-\Delta + \boldsymbol{V}) = [b, +\infty).$$

This shows that the first eigenvalue of $-\Delta + \boldsymbol{V}$ equals $b + \lambda_1$, and the second one is $b + \lambda_2$. By the assumed inequalities (5.72) there holds $-\lambda_2 < b < -\lambda_1$. Therefore, $0 \in \rho(-\Delta + \boldsymbol{V})$ and, as a consequence,

$$\operatorname{Ker}(-\Delta + \boldsymbol{V}) = \{0\}.$$

Furthermore, we have (cf. (2.34) and (2.35) in Example 2.1.15)

$$m_-(-\Delta + \boldsymbol{V}) = \dim \operatorname{Ker}((b + \lambda_1)I - (-\Delta + \boldsymbol{V})) = 1$$

and

$$d = \operatorname{dist}(0, \sigma(-\Delta + \boldsymbol{V}) \cap (0, +\infty)) = b + \lambda_2.$$

Given that $\omega \equiv 0$, we get immediately

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}) = \{0\} \quad \text{and} \quad m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\omega}) = 1.$$

Keeping in mind that $\alpha \equiv g'(0)$, we obtain

$$\sigma_{\text{disc}}(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) = \{\lambda_n + b - g'(0)\}_{n \geq 1} \quad \text{and} \quad \sigma_{\text{ess}}(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) = [b - g'(0), +\infty).$$

From this we deduce that $b + \lambda_1 - g'(0)$ is the first eigenvalue of the operator $-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}$. By the assumption, $g'(0) < b + \lambda_1$, hence, $(-\infty, 0] \subset \rho(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha})$, which yields

$$\operatorname{Ker}(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) = \{0\} \quad \text{and} \quad m_-(-\Delta + \boldsymbol{V} - \boldsymbol{\alpha}) = 0.$$

Let us consider the scalar product $\langle \cdot, \cdot \rangle_{s,V}$ with $s = \beta/3$. Again by (5.72) we obtain that $d = b + \lambda_2 > -\lambda_2 + (\beta/3)^2 + \lambda_2 = (\beta/3)^2$. Therefore, in view of Corollary 2.3.8, the estimate (3.43) is satisfied for $s = \beta/3$ and $\rho > 0$ given by the following formula

$$\rho = s \cdot \frac{1 - \dfrac{s(\beta - s)}{4d}}{1 + \dfrac{s}{\beta - 2s} \cdot \dfrac{s(\beta - s)}{4d}} = (\beta/3) \cdot \frac{1 - \dfrac{(\beta/3)(\beta - \beta/3)}{4d}}{1 + \dfrac{\beta/3}{\beta - 2\beta/3} \cdot \dfrac{(\beta/3)(\beta - \beta/3)}{4d}} = (\beta/3) \cdot \frac{1 - \dfrac{(\beta/3)^2/2}{d}}{1 + \dfrac{(\beta/3)^2/2}{d}}.$$

Considering that $((\beta/3)^2/2)/d < 1/2$ and the function

$$(0, 1) \ni x \mapsto \frac{1 - x}{1 + x}$$

is strictly decreasing, one has

$$\rho > (\beta/3) \cdot \frac{1 - 1/2}{1 + 1/2} = \beta/9.$$

Therefore, from the inequality

$$L_g + L_h\, \widehat{\varrho}(k_\infty) \le (\beta/9)\sqrt{b + \lambda_2}$$

we deduce that

$$\widehat{\varrho}(l_\infty) = L_g + L_h\, \widehat{\varrho}(k_\infty) \le (\beta/9)\sqrt{b + \lambda_2} < \sqrt{d}\, \rho.$$

Since functions ω and α are time-independent, we can use Corollary 5.3.4 to deduce that the *a priori* estimates are satisfied. Then, based on Theorem 5.5.1 (ii) we obtain the existence of a nonzero T-periodic mild solution of (5.70). Thus, the proof is completed. □

We conclude this section with an example of equation (1) with the Coulomb potential and specific classes of nonlinearities to which Remark 5.5.3 applies.

Remark 5.6.4. Let us consider the following equation

$$u_{tt} + \beta u_t = \Delta u + \frac{a}{|x|} u + \lambda_k u + g(t,x)\, b\left(\frac{u}{g(t,x)}\right), \quad t > 0,\ x \in \mathbb{R}^3 \tag{5.75}$$

where $\beta > 0$ is a damping coefficient, $a > 0$ is a constant, and λ_k is the k-th eigenvalue ($k \ge 1$) of the Schrödinger operator (5.71). Function $g : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is T-periodic in time and satisfies Carathéodory condition. Moreover, there exists $g_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ such that

$$0 < g(t,x) \le g_0(x) \quad \text{for almost every}\, x \in \mathbb{R}^3 \text{ and all}\, t \ge 0.$$

Function $b : \mathbb{R} \to \mathbb{R}$ is bounded and satisfies the Lipschitz condition with a constant $L_b < \sqrt{d}\, \rho$. Moreover, b is such that $b(0) = 0$, $b'(0) \ne 0$ and $b'(0) < -\lambda_k$.

Observe that $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, given by

$$V(x) = -\Delta - \frac{a}{|x|} - \lambda_k,$$

is a Kato-Rellich type potential with a positive asymptotic bottom. The nonlinear term f : $[0, +\infty) \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, given by

$$f(t,x,u) = g(t,x)\, b\left(\frac{u}{g(t,x)}\right),$$

is a T-periodic Carathéodory function satisfying condition (f1) with a function $l \equiv L_b$ and condition (f2) with a function $m_0 \equiv 0$. Note that equation (5.75) is at resonance at infinity and f admits a linearization at zero with the coefficient $\alpha \equiv b'(0)$.

(i) Assume that

$$-\lambda_k + \lambda_n < b'(0) < -\lambda_k + \lambda_{n+1} \tag{5.76}$$

where

$$k \equiv 0,\, 3 \pmod 4 \quad \text{and} \quad n \equiv 1,\, 2 \pmod 4$$

or

$$k \equiv 1,\, 2 \pmod 4 \quad \text{and} \quad n \equiv 0,\, 3 \pmod 4. \tag{5.77}$$

Moreover, suppose that

$$b_\pm = \lim_{u \to \pm\infty} b(u) \tag{5.78}$$

are real numbers. If $b_+ > 0$ and $b_- < 0$, then equation (5.75) admits a nonzero T-periodic mild solution. Moreover, the thesis also holds, if $b'(0) < -\lambda_k + \lambda_1$ and $k \ge 1$ is as in (5.77).

(ii) Assume that $b'(0)$ satisfies (5.76), where

$$k \equiv 0,\, 1 \pmod 4 \quad \text{and} \quad n \equiv 1,\, 2 \pmod 4$$

or

$$k \equiv 2,\, 3 \pmod 4 \quad \text{and} \quad n \equiv 0,\, 3 \pmod 4. \tag{5.79}$$

If b_+ and b_- given by (5.78) are real numbers satisfying $b_+ < 0$ and $b_- > 0$, then equation (5.75) admits a nonzero T-periodic mild solution. Moreover, the thesis also holds, if $b'(0) < -\lambda_k + \lambda_1$ and $k \geq 1$ is as in (5.79).

(iii) Assume that $b'(0)$, $k \geq 1$ and $n \geq 1$ are as in the point (i). If there exists $\delta \geq 0$ such that $ub(u) \geq 0$ for all $|u| \geq \delta$ and

$$c_\pm = \lim_{u\to\pm\infty} ub(u) \tag{5.80}$$

are real numbers such that $c_+ > 0$ and $c_- > 0$, then equation (5.75) admits a nonzero T-periodic mild solution. Moreover, the thesis also holds if $b'(0) < -\lambda_k + \lambda_1$ and $k \geq 1$ is as in (5.77).

(iv) Assume that $b'(0)$ satisfies (5.76), where $k \geq 1$ and $n \geq 1$ are as in the point (ii). If there exists $\delta \geq 0$ such that $ub(u) \leq 0$ for all $|u| \geq \delta$ and $c_\pm$ given by (5.80) are real numbers such that $c_+ < 0$ and $c_- < 0$, then equation (5.75) admits a nonzero T-periodic mild solution. Moreover, the thesis also holds if $b'(0) < -\lambda_k + \lambda_1$ and $k \geq 1$ is as in (5.79). □

In the following remark we present specific examples of function b to which equation (5.75) admits a nonzero T-periodic mild solutions.

Remark 5.6.5.

(i) Let us consider function b given by

$$b(u) = \eta \arctan(p(u)) \tag{5.81}$$

where $\eta > 0$ and $p : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $p(0) = 0$, $p'(0) < 0$, $\lim_{u\to+\infty} p(u) = +\infty$, $\lim_{u\to-\infty} p(u) = -\infty$, and

$$C = \sup_{u\in\mathbb{R}} \frac{|p'(u)|}{1 + |p(u)|^2} < +\infty. \tag{5.82}$$

As an example of such function we can take $p(u) = u^3 - u$. We see that function b satisfies $b(0) = 0$, $b'(0) = \eta\, p'(0) < 0$, $b_- = -\eta\,\pi/2$ and $b_+ = \eta\,\pi/2$. Moreover, $L_b = C\eta$.

Assume that $k > 1$ is such that $k \equiv 1,\, 3 \pmod 4$. If $\eta > 0$ is such that

$$\eta < \min\left\{ \frac{-\lambda_k + \lambda_{k-1}}{p'(0)},\ \frac{\sqrt{d}\,\rho}{C} \right\},$$

then, by Remark 5.6.4 (i), equation (5.75) admits a nonzero T-periodic mild solution. Moreover, if $k = 1$ and $\eta < \sqrt{d}\,\rho/C$, then equation (5.75) admits a nonzero T-periodic mild solution.

(ii) Let us consider function b given by (5.81), where $\eta > 0$ and a differentiable function p satisfies $p(0) = 0$, $p'(0) > 0$, $\lim_{u\to+\infty} p(u) = -\infty$, $\lim_{u\to-\infty} p(u) = +\infty$, and p satisfies condition (5.82). As an example of a function p we can take $p(u) = -u^3 + u$. Similarly as in the point (i), we observe that $b(0) = 0$, $b'(0) = \eta\, p'(0) > 0$, $b_- = \eta\,\pi/2$, $b_+ = -\eta\,\pi/2$, and $L_b = C\eta$.

Assume that $k \geq 1$ is such that $k \equiv 1, 3 \pmod 4$. It follows from Remark 5.6.4 (ii) that if

$$\eta < \min\left\{\frac{-\lambda_k + \lambda_{k+1}}{p'(0)}, \frac{\sqrt{d}\,\rho}{C}\right\},$$

then equation (5.75) admits a nonzero T-periodic mild solution.

(iii) We define function b as follows

$$b(u) = \begin{cases} \eta\, p(u) & \text{if } |u| < 1, \\ \eta\, u/(1+u^2) & \text{if } |u| \geq 1, \end{cases}$$

where $\eta > 0$, $p : [-1, 1] \to \mathbb{R}$ satisfies the Lipschitz condition with a constant denoted by $L_p > 0$, and is such that $p(0) = 0$, $p'(0) < 0$, $p(1) = 1/2$, and $p(-1) = -1/2$. Function $p(u) = u^3 - (1/2)u$ serves as an example of such function. Observe that function b satisfies $b(0) = 0$, $b'(0) = \eta\, p'(0) < 0$, $c_+ = c_- = \eta$, and $ub(u) \geq 0$ for all $|u| \geq 1$. Moreover, we have $L_b = \eta\, D$, where

$$D = \max\left\{L_p, \sup_{|u|\geq 1} \frac{|1-u^2|}{|1+u^2|^2}\right\}. \tag{5.83}$$

If $k > 1$ satisfies $k \equiv 1, 3 \pmod 4$ and $\eta > 0$ is such that

$$\eta < \min\left\{\frac{-\lambda_k + \lambda_{k-1}}{p'(0)}, \frac{\sqrt{d}\,\rho}{D}\right\},$$

then, by Remark 5.6.4 (iii), equation (5.75) admits a nonzero T-periodic mild solution. In addition, the thesis also holds if $k = 1$ and $\eta < \sqrt{d}\,\rho/D$.

(iv) Let us consider function b given by

$$b(u) = \begin{cases} -\eta\, p(u) & \text{if } |u| < 1, \\ -\eta\, u/(1+u^2) & \text{if } |u| \geq 1, \end{cases}$$

where $\eta > 0$ is arbitrary and function $p : [-1, 1] \to \mathbb{R}$ is as in the point (iii). It follows from the point (iii) that function b satisfies $b(0) = 0$, $b'(0) = -\eta\, p'(0) > 0$, $c_+ = c_- = -\eta$, and $ub(u) \leq 0$ for all $|u| \geq 1$. Moreover, $L_b = \eta\, D$, where $D > 0$ is given by (5.83).

If $k \geq 1$ satisfies $k \equiv 1, 3 \pmod 4$ and $\eta > 0$ satisfies

$$\eta < \min\left\{\frac{-\lambda_k + \lambda_{k+1}}{-p'(0)}, \frac{\sqrt{d}\,\rho}{D}\right\},$$

then, by Remark 5.6.4 (iv), there exists a nonzero T-periodic mild solution of equation (5.75). □

Appendix

A.1 Linear operators on Banach spaces

We start by recalling the notion of complexification, which is needed to consider the complex spectrum and resolvent set for linear operators on spaces over real numbers. Let $(X, \|\cdot\|)$ be a real normed space. We define the complexificated space $X_{\mathbb{C}}$ as $X_{\mathbb{C}} = X \times X$. We denote $X \oplus iX = X_{\mathbb{C}}$, $U \oplus iV = U \times V$, for $U \times V \subset X_{\mathbb{C}}$, and $u \oplus iv = (u, v)$, for $(u, v) \in X_{\mathbb{C}}$. Sum of two vectors $u_1 \oplus iv_1$, $u_2 \oplus iv_2$ is given by

$$(u_1 \oplus iv_1) + (u_2 \oplus iv_2) = (u_1 + u_2) \oplus i(v_1 + v_2)$$

and multiplication of a vector $u \oplus iv$ by a scalar $\lambda + i\mu \in \mathbb{C}$ is given by

$$(\lambda + i\mu)(u \oplus iv) = (\lambda u - \mu v) \oplus i(\mu u + \lambda v).$$

We define a norm on the space $X_{\mathbb{C}}$ as

$$\|u \oplus iv\|_{\mathbb{C}} = \sup_{\varphi \in [0,2\pi]} \|\sin\varphi\, u + \cos\varphi\, v\| \quad \text{for } u \oplus iv \in X_{\mathbb{C}}.$$

The real space X embeds in $X_{\mathbb{C}}$ in the standard way: $u \mapsto u \oplus i0$, for $u \in X$. One can show that if $(X, \|\cdot\|)$ is a Banach space, then $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ is also a Banach space.

If $(X, \langle\cdot,\cdot\rangle)$ is a unitary space with the induced norm $\|\cdot\|$, we define a scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on $X_{\mathbb{C}}$ as follows:

$$\langle u_1 \oplus iv_1, u_2 \oplus iv_2\rangle_{\mathbb{C}} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle + i\big(-\langle u_1, v_2\rangle + \langle v_1, u_2\rangle\big).$$

The scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ generates the norm $\|\cdot\|_{\mathbb{C}}$ on $X_{\mathbb{C}}$:

$$\|u \oplus iv\|_{\mathbb{C}}^2 = \|u\|^2 + \|v\|^2 (^1).$$

If $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space, then $(X_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ is also a Hilbert space.

Let $A : D(A) \subset X \to X$ be a closed linear operator on a real Banach space X. We define the complexificated operator $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \subset X_{\mathbb{C}} \to X_{\mathbb{C}}$

$$A_{\mathbb{C}}(u \oplus iv) = Au \oplus iAv \quad \text{for} \quad u \oplus iv \in D(A_{\mathbb{C}}) = D(A) \oplus iD(A).$$

The real resolvent set of A, $\rho_{\mathbb{R}}(A)$, consists of $\lambda \in \mathbb{R}$ such that $\lambda I - A : D(A) \subset X \to X$ is invertible and $(\lambda I - A)^{-1} : X \to X$ is a bounded linear operator. If $\lambda \in \rho_{\mathbb{R}}(A)$, then $R(\lambda, A) = (\lambda I - A)^{-1}$. Next, the real spectrum of A is given by $\sigma_{\mathbb{R}}(A) = \mathbb{R} \setminus \rho_{\mathbb{R}}(A)$. Further, we define the resolvent set of A and the spectrum of A:

$$\rho(A) = \rho(A_{\mathbb{C}}) \ \text{ and } \ \sigma(A) = \sigma(A_{\mathbb{C}})$$

where, as in the real case, $\rho(A_{\mathbb{C}})$ consists of $\lambda \in \mathbb{C}$ such that $\lambda I - A_{\mathbb{C}} : D(A_{\mathbb{C}}) \subset X_{\mathbb{C}} \to X_{\mathbb{C}}$ is invertible and $(\lambda I - A_{\mathbb{C}})^{-1} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is a bounded linear operator. Moreover, $R(\lambda, A_{\mathbb{C}}) = (\lambda I - A_{\mathbb{C}})^{-1}$ for $\lambda \in \rho(A_{\mathbb{C}})$.

The result below provides the basic properties of the complexificated operators and relationship between spectrum, resolvent set, resolvent operator and their real counterparts.

(^{1})Note that this map does not need to be a norm on $X_{\mathbb{C}}$ if X is a general normed space.

Proposition A.1.1. [see [4, pp. 4-5]] *Assume that $A : D(A) \subset X \to X$ is a closed linear operator on a real Banach space X.*

(i) *The linear operator $A_{\mathbb{C}}$ is also closed. Moreover, $D(A_{\mathbb{C}})$ is a dense subset of $X_{\mathbb{C}}$ if and only if $D(A)$ is a dense subset of X. Assuming that X is a Hilbert space, $A_{\mathbb{C}}$ is a self-adjoint operator if and only if A is self-adjoint.*

(ii) *$\sigma_{\mathbb{R}}(A) = \sigma(A) \cap \mathbb{R}$ and $\rho_{\mathbb{R}}(A) = \rho(A) \cap \mathbb{R}$, and if $\lambda \in \rho(A) \cap \mathbb{R}$, then $R(\lambda, A_{\mathbb{C}})|_X = R(\lambda, A)$.*

Let $A : D(A) \subset X \to X$ be a closed linear operator. The *point spectrum* of A is defined as

$$\sigma_p(A) = \big\{\lambda \in \mathbb{C} \ : \ \mathrm{Ker}(\lambda I - A) \neq \{0\}\big\}.$$

Any element $\lambda \in \sigma_p(A)$ is called *eigenvalue* of A and each $u \in \mathrm{Ker}(\lambda I - A) \setminus \{0\}$ is called *eigenvector* corresponding to λ. For any eigenvalue of operator A we define its *geometric* and *algebraic multiplicity*:

$$m_{\mathrm{geo}}(\lambda) = \dim \mathrm{Ker}(\lambda I - A) \ \ \text{and} \ \ m_{\mathrm{alg}}(\lambda) = \dim \bigcup_{n=1}^{\infty} \mathrm{Ker}(\lambda I - A)^n.$$

In general, geometric and algebraic multiplicities are different, however, if A is a self-adjoint operator on Hilbert space X, there holds $m_{\mathrm{geo}}(\lambda) = m_{\mathrm{alg}}(\lambda)$, for all $\lambda \in \sigma_p(A)$. Further, the *discrete spectrum* is given by

$$\sigma_{\mathrm{disc}}(A) = \big\{\lambda \in \sigma_p(A) \ : \ \exists_{r>0} \ B_{\mathbb{C}}(\lambda, r) \setminus \{\lambda\} \subset \rho(A) \ \ \text{and} \ \ m_{\mathrm{alg}}(\lambda) < \infty\big\}.$$

In other words, the discrete spectrum consists of the isolated eigenvalues with finite algebraic multiplicities. Finally, the *essential spectrum* is simply the complement of the discrete spectrum in $\sigma(A)$, i.e., $\sigma_{\mathrm{ess}}(A) = \sigma(A) \setminus \sigma_{\mathrm{disc}}(A)$.

Next result collects some auxiliary facts concerning decompositions of Banach spaces.

Proposition A.1.2. *Let X be a Banach space such that $X = X_P \oplus X_Q$ with projections P, Q onto X_P, X_Q, respectively.*

(i) *If X_P, X_Q are closed, then projections P, Q are continuous.*

(ii) *Assume that X_P, X_Q are closed subspaces of X, Y is a continuously embedded subspace of X and X_P is a subspace of Y. Then $Y = X_P \oplus (Y \cap X_Q)$ and X_P, $Y \cap X_Q$ are closed subspaces of Y.*

(iii) *If X_P, X_Q are closed subspaces of X, Y is a subspace of X and X_P is a subspace of Y, then $\overline{Y \cap X_Q}^X = \overline{Y}^X \cap X_Q$.*

Proof. (i) This result can be found in [9, Thm. 2.10].

(ii) Let $x \in Y$. Then there exist unique $x_P \in X_P$, $x_Q \in X_Q$ such that $x = x_P + x_Q$. Since X_P is a subspace of Y, one has $x_Q \in Y \cap X_Q$. Hence, $Y = X_P \oplus (Y \cap X_Q)$. Suppose that $(x_n)_{n\geq 1}$ is a sequence in X_P such that $x_n \to x_0$ in Y. This implies that $x_n \to x_0$ in X. Therefore, because X_P is a closed subspaces of X, we deduce that $x_0 \in X_P$. In a similar manner one can show that $Y \cap X_Q$ is a closed subspace of Y.

(iii) Obviously, because X_Q is closed, $\overline{Y \cap X_Q}^X \subset \overline{Y}^X \cap X_Q$. Next, we take $x_0 \in \overline{Y}^X \cap X_Q$. Then we have a sequence $(x_n)_{n\geq 1}$ in Y such that $x_n \to x_0$ in X. Note that, because X_P is a subspace of Y, $Qx_n \in Y \cap X_Q$, for any $n \geq 1$. By the point (i), Q is continuous, thus $Qx_n \to Qx_0 = x_0$. Hence, $\overline{Y}^X \cap X_Q \subset \overline{Y \cap X_Q}^X$. □

Now, we recall several results from the perturbation theory of linear operators.

Definition A.1.3. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be linear operators on Banach space $(X, \|\cdot\|)$.

(i) Operator B is said to be *(relatively) A-bounded* if $D(A) \subset D(B)$ and the linear operator

$$\widetilde{B} : (D(A), \|\cdot\|_A) \to (X, \|\cdot\|), \ \widetilde{B}u = Bu \text{ for } u \in D(A), \tag{A.1}$$

is bounded, where $\|u\|_A = (\|u\|^2 + \|Au\|^2)^{1/2}$ is the graph norm on $D(A)$. If B is A-bounded, then the number

$$\inf\{b \geq 0 \ : \text{there exists } a \geq 0 \text{ such that } \|Bu\| \leq a\|u\| + b\|Au\| \text{ for all } u \in D(A)\}$$

is called the A-bound of the operator B.

(ii) We say that B is *(relatively) A-compact* if $D(A) \subset D(B)$ and the linear operator $\widetilde{B}$, given by (A.1), is *compact*, that is, set $\widetilde{B}(U)$ is relatively compact, for any bounded $U \subset D(A)$, where $D(A)$ is understood as the normed space with the graph norm $\|\cdot\|_A$.

Theorem A.1.4. [see [58, Thm. 9.7]] *Assume that operator B is relatively A-compact, where $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ are linear operators on a Hilbert space X. If A or B is closable*(2)*, then B is A-bounded with the zero A-bound.*

Theorem A.1.5. *Let $A : D(A) \subset X \to X$ be a self-adjoint operator on a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ with induced norm $\|\cdot\|$. Then*

(i) [36, Prop. 4.9, Thm. 5.5, Prop. 5.12] *A is closed, $\sigma(A) \subset \mathbb{R}$ and, for any $a \in \mathbb{R}$, $\sigma(A) \subset [a, +\infty)$ if and only if, for all $u \in D(A)$, we have $\langle Au, u\rangle \geq a\|u\|^2$.*

(ii) [Kato-Rellich Theorem, see [58, Thm. 5.28]] *Let $B : D(B) \subset X \to X$ be a symmetric operator which is A-bounded with A-bound less then one. Then the operator $A+B$ on the domain $D(A+B) = D(A)$ is also self-adjoint. In particular, if $B : X \to X$ is a bounded symmetric operator, then $A + B$ is self-adjoint.*

(iii) [Weyl's Theorem, see [58, Thm. 9.9]] *Suppose that $B : D(B) \subset X \to X$ is a symmetric operator which is relatively A-compact. Then $\sigma_{\text{ess}}(A + B) = \sigma_{\text{ess}}(A)$.*

Now we present the spectral theorem for self-adjoint operators.

Theorem A.1.6. (see [34, Thm. 10.4] and [55, Thm. 5.7]) *Let $A : D(A) \subset X \to X$ be the self-adjoint operator on a Hilbert space X such that $\sigma(A) = \sigma_P \cup \sigma_Q$, where σ_P, σ_Q are disjoint sets, σ_P is compact and σ_Q is closed.*

Then X admits the unique orthogonal decomposition $X = X_P \oplus X_Q$, X_j, $j \in \{P, Q\}$, are invariant under A, $A|_{X_P}$ is bounded and self-adjoint, $A|_{X_Q}$ is self-adjoint, and $\sigma\left(A|_{X_j}\right) = \sigma_j$, $j \in \{P, Q\}$. In addition, if σ_P is finite, then $\sigma_P \subset \sigma_{\text{disc}}(A)$ and $X_P = \bigoplus_{\lambda\in\sigma_P} \operatorname{Ker}(\lambda I - A)$.

Proposition A.1.7. *Let $A : D(A) \subset X \to X$ be a closed operator in a Banach space X and $B : X \to X$ be a bounded and invertible linear operator such that $B(D(A)) \subset D(A)$. Then the linear operator $BAB^{-1} : D(A) \subset X \to X$ is well-defined and satisfies*

$$\sigma_{\text{disc}}(BAB^{-1}) = \sigma_{\text{disc}}(A) \quad \text{and} \quad \sigma_{\text{ess}}(BAB^{-1}) = \sigma_{\text{ess}}(A).$$

Additionally,

$$\dim \operatorname{Ker}\big(\lambda I - BAB^{-1}\big) = \dim \operatorname{Ker}(\lambda I - A) \quad \text{for any } \lambda \in \sigma_{\text{disc}}(A).$$

(2)Recall that linear operator $A : D(A) \subset X \to X$ on a Banach space X is said to be closable if $\overline{\operatorname{Gr} A}$, the closure of its graph, is a graph of a certain linear operator $B : D(B) \subset X \to X$ (cf. [58, Section 5.1]).

A.2 Basic inequalities and function spaces

In the following lemma we provide some useful inequalities.

Lemma A.2.1.

(i) [Young's inequality] *Let* $\varepsilon > 0$ *and* p_1, $p_2 > 1$ *be such that* $1/p_1 + 1/p_2 = 1$. *Then we have*

$$ab \le \frac{a^{p_1}}{p_1 \varepsilon^{p_1 - 1}} + \frac{\varepsilon b^{p_2}}{p_2} \quad \text{for all } a, b \ge 0.$$

(ii) [Gronwall's inequality] *Assume that* a, b, $c : [0, T_0] \to \mathbb{R}$ *are continuous functions such that* $b(t)$, $c(t) \ge 0$, *for all* $t \in [0, T_0]$. *If a continuous function* $f : [0, T_0] \to \mathbb{R}$ *satisfies*

$$f(t) \le a(t) + b(t) \int_0^t c(\tau) f(\tau)\, d\tau \quad \text{for all } t \in [0, T_0],$$

then

$$f(t) \le a(t) + b(t) \int_0^t a(\tau) c(\tau) \exp\left(\int_\tau^t b(\sigma) c(\sigma)\, d\sigma \right) d\tau \quad \text{for all } t \in [0, T_0].$$

In particular, if $b(t) = 1$ *and* $c(t) = c_0$, *for all* $t \in [0, T_0]$, *where* $c_0 \ge 0$, *then*

$$f(t) \le a(t) + c_0 \int_0^t a(\tau) e^{c_0 (t - \tau)}\, d\tau \quad \text{for all } t \in [0, T_0],$$

and if $a(t) = a_0$, $b(t) = 1$, $c(t) = c_0$, *for all* $t \in [0, T_0]$, *where* $a_0 \in \mathbb{R}$ *and* $c_0 \ge 0$, *then*

$$f(t) \le a_0 e^{c_0 t} \quad \text{for all } t \in [0, T_0].$$

The following result collects some facts about Lebesgue spaces which are used within this work.

Proposition A.2.2.

(i) [Interpolation inequality, see [9, Exercise 4.4]] *Let* $U \subset \mathbb{R}^N$ *be a measurable set and* $f \in L^p(U) \cap L^q(U)$, *for some* $1 \le p \le q \le +\infty$. *Then, the function* f *belongs to the space* $L^r(U)$, *for each* $r \in [p, q]$, *and moreover*

$$\|f\|_{L^r(U)} \le \|f\|_{L^p(U)}^{\theta}\ \|f\|_{L^q(U)}^{1-\theta}$$

where number $\theta \in [0, 1]$ *satisfies the equation*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1 - \theta}{q}.$$

(ii) [see [9, Thm. 4.9]] *If* $(f_n)_{n \ge 1}$ *is a sequence such that* $f_n \to f$ *in* $L^p(\mathbb{R}^N)$, *where* $1 \le p \le +\infty$, *then there exists a subsequence* $(f_{n_k})_{k \ge 1}$ *and a function* $h \in L^p(\mathbb{R}^N)$ *which satisfy*

- $f_{n_k}(x) \to f(x)$, *for almost every* $x \in \mathbb{R}^N$,
- $|f_{n_k}(x)| \le h(x)$, *for all* $k \ge 1$ *and almost every* $x \in \mathbb{R}^N$.

(iii) [see [9, Thm. 4.12]] *The space* $C_0^\infty(\mathbb{R}^N)$ *is dense in* $L^p(\mathbb{R}^N)$, *for all* $1 \le p < +\infty$.

Below, we present several results concerning Sobolev spaces.

Theorem A.2.3. [Friedrich's Theorem, see [9, Thm. 9.2]] *If* $u \in W^{1,p}(\mathbb{R}^N)$*, where* $p \in [1, +\infty)$*, then there exists a sequence* $(u_n)_{n\geq 1}$ *in* $C_0^\infty(\mathbb{R}^N)$ *such that*

$$u_n \to u \text{ in } W^{1,p}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Theorem A.2.4. [see [9, Corollary 9.13 and Corollary 9.15]] *Let* $p \in [1, +\infty)$*,* $k \geq 1$ *be some integer and* $U = \mathbb{R}^N$ *or* $U \subset \mathbb{R}^N$ *is an open set of class* C^1(3) *with bounded boundary, and the number* q *is such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{N}.$$

Then the following embeddings are continuous:

$$\begin{array}{lll} W^{k,p}(U) \hookrightarrow L^r(U) & \text{for } r \in [p, q] & \text{if } \frac{1}{p} - \frac{k}{N} > 0, \\ W^{k,p}(U) \hookrightarrow L^r(U) & \text{for } r \in [p, +\infty) & \text{if } \frac{1}{p} - \frac{k}{N} = 0, \\ W^{k,p}(U) \hookrightarrow L^r(U) & \text{for } r \in [p, +\infty] & \text{if } \frac{1}{p} - \frac{k}{N} < 0. \end{array}$$

Theorem A.2.5. [Rellich-Kondrachov Theorem, see [9, Thm. 9.16]] *We assume that* $U \subset \mathbb{R}^N$ *is an open bounded subset of class* C^1 *and* $p \in [1, +\infty]$*. Then, the following embeddings are compact:*

$$\begin{array}{lll} W^{1,p}(U) \hookrightarrow L^r(U) & \text{for } r \in [1, q) & \text{if } \frac{1}{q} = \frac{1}{p} - \frac{1}{N} > 0, \\ W^{1,p}(U) \hookrightarrow L^r(U) & \text{for } r \in [1, +\infty) & \text{if } \frac{1}{p} - \frac{1}{N} = 0, \\ W^{1,p}(U) \hookrightarrow C(\overline{U}) & & \text{if } \frac{1}{p} - \frac{1}{N} < 0. \end{array}$$

The compact embedding $W^{1,p}(U) \hookrightarrow C(\overline{U})$ *is understood modulo the choice of a continuous representative. In particular, for all* $p \in [1, +\infty]$ *and all* $N \geq 1$*, the embedding* $W^{1,p}(U) \hookrightarrow L^p(U)$ *is compact.*

A.3 Measure of non-compactness

Assume that $(X, \|\cdot\|)$ is a Banach space. The map $\chi : \mathfrak{B} \to [0, +\infty)$, where $\mathfrak{B}$ denotes the set of bounded subsets of X, defined as

$$\chi(U) = \inf\{r > 0 : U \text{ admits a finite cover by open balls of radius } r \\ \text{which centers belong to } X\},$$

is called the *Hausdorff measure of non-compactness.*(4) Now, we list its basic properties.

Proposition A.3.1. [see [1, Sec. 1.1]] *Suppose that* U*,* U_1 *and* $U_2 \subset X$ *are bounded. The following claims are true:*

(i) (regularity) $\chi(U) = 0$ *if and only if* U *is relatively compact;*

(ii) (invariance under passage to the closure and to the convex hull) $\chi(U) = \chi(\overline{U}) = \chi(\operatorname{conv} U)$*;*

(iii) (algebraic semi-additivity) $\chi(U_1 + U_2) \leq \chi(U_1) + \chi(U_2)$*, where* $U_1 + U_2 = \{x_1 + x_2 : x_1 \in U_1,\ x_2 \in U_2\}$*;*

(3)For the definition of sets of class C^1 see [9, p. 272]
(4)One can also consider the *Kuratowski measure of non-compactness* $\gamma : \mathfrak{B} \to [0, +\infty)$ given by

$$\gamma(U) = \inf\{d > 0 : U \text{ admits a finite cover by sets of diameter no larger than } d\}.$$

(iv) (semi-homogencity) *for any fixed* $\lambda \in \mathbb{R}$, $\chi(\lambda U) = |\lambda|\chi(U)$, *where* $\lambda U = \{\lambda x : x \in U\}$, *and, if* $D \subset \mathbb{R}$ *is bounded, then*

$$\chi(\{\lambda \cdot u : \lambda \in D,\, u \in U\}) \le \Big(\sup_{\lambda \in D} |\lambda|\Big)\chi(U);$$

(v) *if* $\Phi : X \to X$ *satisfies the Lipschitz condition with a constant* $L > 0$, *then* $\chi(\Phi(U)) \le L\chi(U)$; *in particular, if* $A : X \to X$ *is a bounded linear operator,* $\chi(A(U)) \le \|A\|_{\mathcal{L}(X)}\chi(U)$;

(vi) (monotonicity) *if* $U_1 \subset U_2$, *then* $\chi(U_1) \le \chi(U_2)$;

(vii) (semi-additivity) $\chi(U_1 \cup U_2) = \max\{\chi(U_1), \chi(U_2)\}$;

(viii) *assume that* X_1 *is a closed subspace of* X *and consider the Banach space* X_1 *with an induced norm and measure of non-compactness* χ_1; *then* $\chi(U) \le \chi_1(U)$, *for each bounded subset* $U \subset X_1$; *by the additional assumption that* X *is a Hilbert space, we have* $\chi(U) = \chi_1(U)$.

Lemma A.3.2. [compare with [17, Lemma 5.5]] *We assume that* $L_n : X \to X$, $n \ge 1$, *is a sequence of bounded linear operators such that* $(\|L_n\|_{\mathcal{L}(X)})_{n\ge1}$ *is bounded, and, for any* $x \in X$, *the set* $\{L_n x\}_{n\ge1}$ *is relatively compact. Then, for any bounded sequence* $(x_n)_{n\ge1}$ *in* X,

$$\chi(\{L_n x_n\}_{n\ge1}) \le \Big(\limsup_{n\to\infty} \|L_n\|_{\mathcal{L}(X)}\Big)\chi(\{x_n\}_{n\ge1}).$$

The next important result provides a rule allowing to interchange integral and measure of non-compactness.

Theorem A.3.3. [see [22, Proposition 9.3]] *Assume that* X *is a separable Banach space and* $f_n : [0, T_0] \to X$, $n \ge 1$, *is a uniformly bounded sequence of continuous functions, where* $0 < T_0 < +\infty$. *Then,*

(i) *function*

$$[0, T_0] \ni t \mapsto \chi(\{f_n(t) : n \ge 1\}) \in \mathbb{R}$$

is measurable and bounded;

(ii) *for all* $0 \le a \le b \le T_0$,

$$\chi\left(\left\{\int_a^b f_n(t)\,dt : n \ge 1\right\}\right) \le \int_a^b \chi(\{f_n(t) : n \ge 1\})\,dt.$$

A.4 Topological degree and index

In this appendix, we present the theory of the topological degree and index for various classes of mappings. We begin by the Brouwer degree and index for continuous mappings in finite-dimensional spaces.

Let X be a finite-dimensional normed space and $U \subset X$ be an open bounded subset. We say that a map $\Phi : \overline{U} \to X$ is *admissible for the Brouwer degree*, if Φ is continuous and $\Phi(x) \ne 0$ for all $x \in \partial U$.

Theorem A.4.1. [see [24, Ch. IV, §10, Thm. 8.5]]
For any map $\Phi : \overline{U} \to X$ *that is admissible for the Brouwer degree, there exists an integer* $\deg_B(\Phi, U)$, *called the* Brouwer topological degree, *which satisfies the following properties:*

(D1) *(Normalization) If* $\Phi(x) = x$ *for all* $x \in \overline{U}$ *and* $0 \in U$, *then* $\deg_B(\Phi, U) = 1$.

(D2) *(Additivity) If open disjoint sets* U_1, $U_2 \subset U$ *are such that* $\Phi(x) \neq 0$ *for* $x \in U \setminus (U_1 \cup U_2)$*, then*

$$\deg_B(\Phi, U) = \deg_B(\Phi|_{\overline{U}_1}, U_1) + \deg_B(\Phi|_{\overline{U}_2}, U_2).$$

(D3) *(Homotopy invariance) If a map* $H : \overline{U} \times [0,1] \to X$ *is an* admissible homotopy for the Brouwer degree*, i.e.,* H *is continuous and* $H(x, \mu) \neq 0$ *for all* $x \in \partial U$ *and* $\mu \in [0,1]$*, then*

$$\deg_B(H(\cdot, 0), U) = \deg_B(H(\cdot, 1), U).$$

(D4) *(Existence) If* $\deg_B(\Phi, U) \neq 0$*, then there exists* $x \in U$ *such that* $\Phi(x) = 0$*.*

(D5) *(Excision) If an open set* $U_1 \subset U$ *is such that* $\Phi(x) \neq 0$ *for* $x \in U \setminus U_1$*, then* $\deg_B(\Phi, U) = \deg_B(\Phi|_{\overline{U}_1}, U_1)$*.*

(D6) *(Contraction) If* $\Phi(\overline{U}) \subset X_0$*, where* X_0 *is a linear subspace of* X*, then*

$$\deg_B(\Phi, U) = \deg_B\left(\Phi|_{\overline{U} \cap X_0}, U \cap X_0\right).$$

(D7) *(Multiplicativity) If maps* $\Phi_1 : \overline{U}_1 \to X_1$ *and* $\Phi_2 : \overline{U}_2 \to X_2$ *are admissible for the Brouwer degree, then the map* $\Phi_1 \times \Phi_2 : \overline{U}_1 \times \overline{U}_2 \to X_1 \times X_2$*, defined by*

$$(\Phi_1 \times \Phi_2)(x_1, x_2) = (\Phi_1(x_1), \Phi_2(x_2)), \tag{A.2}$$

is also admissible, and we have

$$\deg_B(\Phi_1 \times \Phi_2, U_1 \times U_2) = \deg_B(\Phi_1, U_1) \cdot \deg_B(\Phi_2, U_2).$$

Now, we present the theory of the *Brouwer topological index.* As before, let X be a finite-dimensional normed space and let $U \subset X$ be an open bounded set. We say that a map $\Phi : \overline{U} \to X$ is *admissible for the Brouwer index*, if Φ is continuous and $\Phi(x) \neq x$ for $x \in \partial U$. Observe that

$$\mathrm{Fix}(\Phi, U) = \{x \in U : \Phi(x) = x\} = \{x \in U : (\mathrm{Id} - \Phi)x = 0\}.$$

Therefore, for any admissible map $\Phi : \overline{U} \to X$, we define an integer $\mathrm{Ind}_B(\Phi, U)$, called the *Brouwer topological index* by the following formula:

$$\mathrm{Ind}_B(\Phi, U) = \deg_B(\mathrm{Id} - \Phi, U). \tag{A.3}$$

Theorem A.4.2. [see [24, Ch. IV, §12, Thm. 1.2]]
For any map $\Phi : \overline{U} \to X$ *that is admissible for the Brouwer index, the integer* $\mathrm{Ind}_B(\Phi, U)$*, given by* (A.3)*, is well-defined and satisfies the following properties:*

(I1) *(Normalization) If* $\Phi(x) = x_0$ *for all* $x \in \overline{U}$*, with* $x_0 \in U$*, then* $\mathrm{Ind}_B(\Phi, U) = 1$*.*

(I2) *(Additivity) If open disjoint sets* U_1, $U_2 \subset U$ *are such that* $\Phi(x) \neq x$ *for* $x \in U \setminus (U_1 \cup U_2)$*, then*

$$\mathrm{Ind}_B(\Phi, U) = \mathrm{Ind}_B(\Phi|_{\overline{U}_1}, U_1) + \mathrm{Ind}_B(\Phi|_{\overline{U}_2}, U_2).$$

(I3) *(Homotopy invariance) If a map* $H : \overline{U} \times [0,1] \to X$ *is an* admissible homotopy for the Brouwer index*, i.e.,* H *is continuous and* $H(x, \mu) \neq x$ *for all* $x \in \partial U$ *and* $\mu \in [0,1]$*, then*

$$\mathrm{Ind}_B(H(\cdot, 0), U) = \mathrm{Ind}_B(H(\cdot, 1), U).$$

(I4) *(Existence) If* $\mathrm{Ind}_B(\Phi, U) \neq 0$*, then there exists* $x \in U$ *such that* $\Phi(x) = x$*.*

(I5) *(Excision) If an open set* $U_1 \subset U$ *is such that* $\Phi(x) \neq x$ *for* $x \in U \setminus U_1$*, then* $\mathrm{Ind}_B(\Phi, U) = \mathrm{Ind}_B(\Phi|_{\overline{U}_1}, U_1)$.

(I6) *(Contraction) If* $\Phi(\overline{U}) \subset X_0$*, where* X_0 *is a linear subspace of* X*, then*

$$\mathrm{Ind}_B(\Phi, U) = \mathrm{Ind}_B\left(\Phi|_{\overline{U}\cap X_0}, U \cap X_0\right).$$

(I7) *(Multiplicativity) If maps* $\Phi_1 : \overline{U}_1 \to X_1$ *and* $\Phi_2 : \overline{U}_2 \to X_2$ *are admissible for the Brouwer index, then the map* $\Phi_1 \times \Phi_2 : \overline{U}_1 \times \overline{U}_2 \to X_1 \times X_2$*, defined by* (A.2)*, is also admissible, and we have*

$$\mathrm{Ind}_B(\Phi_1 \times \Phi_2, U_1 \times U_2) = \mathrm{Ind}_B(\Phi_1, U_1) \cdot \mathrm{Ind}_B(\Phi_2, U_2).$$

We now present the theory of the *Leray-Schauder topological index.* Let X and Y be normed spaces, not necessarily finite-dimensional, and let $V \subset X$ be an arbitrary subset, not necessarily open or bounded. A map $\Phi : V \to Y$ is said to be *completely continuous* if Φ is continuous and the image $\Phi(W)$ is relatively compact for every bounded subset $W \subset V$. In addition, a completely continuous linear map $L : X \to Y$, defined on the whole space X, is called a *compact operator.*

Now, suppose that $U \subset X$ is an open bounded subset. We say that a map $\Phi : \overline{U} \to X$ is *admissible for the Leray-Schauder index*, if Φ is completely continuous and $\Phi(x) \neq x$ for all $x \in \partial U$. Moreover, we say that a map $H : \overline{U} \times [0,1] \to X$ is an *admissible homotopy for the Leray-Schauder index*, if H is continuous, the set $H(\overline{U} \times [0,1])$ is relatively compact, and $H(x, \mu) \neq x$ for all $x \in \partial U$ and $\mu \in [0,1]$.

Theorem A.4.3. [see [24, Ch. IV, §12, Thm. 3.4]]
For any map $\Phi : \overline{U} \to X$ *that is admissible for the Leray-Schauder index, there exists an integer* $\mathrm{Ind}_{LS}(\Phi, U)$*, called the* Leray-Schauder topological index*, which satisfies the following properties:*

(LS1) *(Normalization) If* $\Phi(x) = x_0$ *for all* $x \in \overline{U}$*, with* $x_0 \in U$*, then* $\mathrm{Ind}_{LS}(\Phi, U) = 1$.

(LS2) *(Additivity) If open disjoint sets* $U_1, U_2 \subset U$ *are such that* $\Phi(x) \neq x$ *for* $x \in U \setminus (U_1 \cup U_2)$*, then*

$$\mathrm{Ind}_{LS}(\Phi, U) = \mathrm{Ind}_{LS}(\Phi|_{\overline{U}_1}, U_1) + \mathrm{Ind}_{LS}(\Phi|_{\overline{U}_2}, U_2).$$

(LS3) *(Homotopy invariance) If a map* $H : \overline{U} \times [0,1] \to X$ *is an admissible homotopy for the Leray-Schauder index, then*

$$\mathrm{Ind}_{LS}(H(\cdot, 0), U) = \mathrm{Ind}_{LS}(H(\cdot, 1), U).$$

(LS4) *(Existence) If* $\mathrm{Ind}_{LS}(\Phi, U) \neq 0$*, then there exists* $x \in U$ *such that* $\Phi(x) = x$.

(LS5) *(Excision) If an open set* $U_1 \subset U$ *is such that* $\Phi(x) \neq x$ *for* $x \in U \setminus U_1$*, then* $\mathrm{Ind}_{LS}(\Phi, U) = \mathrm{Ind}_{LS}(\Phi|_{\overline{U}_1}, U_1)$.

(LS6) *(Contraction) If* $\Phi(\overline{U}) \subset X_0$*, where* X_0 *is a closed subspace of* X*, then*

$$\mathrm{Ind}_{LS}(\Phi, U) = \mathrm{Ind}_{LS}\left(\Phi|_{\overline{U}\cap X_0}, U \cap X_0\right).$$

(LS7) *(Multiplicativity) If maps* $\Phi_1 : \overline{U}_1 \to X_1$ *and* $\Phi_2 : \overline{U}_2 \to X_2$ *are admissible for the Leray-Schauder index, then the map* $\Phi_1 \times \Phi_2 : \overline{U}_1 \times \overline{U}_2 \to X_1 \times X_2$*, defined by* (A.2)*, is also admissible, and we have*

$$\mathrm{Ind}_{LS}(\Phi_1 \times \Phi_2, U_1 \times U_2) = \mathrm{Ind}_{LS}(\Phi_1, U_1) \cdot \mathrm{Ind}_{LS}(\Phi_2, U_2).$$

(LS8) *(Topological invariance)* [see [31, property (IX) on p. 217]] *Assume that* Y *is a normed space and that* $h : X \to Y$ *is a homeomorphism. Then the map* $h \circ \Phi \circ h^{-1} : \overline{h(U)} \to Y$ *is admissible for the Leray-Schauder index, and*

$$\mathrm{Ind}_{LS,X}(\Phi, U) = \mathrm{Ind}_{LS,Y}(h \circ \Phi \circ h^{-1}, h(U))$$

where $\mathrm{Ind}_{LS,X}(\cdot,\cdot)$ *denotes the Leray-Schauder index in the space* X*, and* $\mathrm{Ind}_{LS,Y}(\cdot,\cdot)$ *denotes the Leray-Schauder index in* Y*.*

Moreover, if the space X *is finite-dimensional, then* $\mathrm{Ind}_{LS}(\Phi, U) = \mathrm{Ind}_B(\Phi, U)$*.*

Next result provides the Leray-Schauder index formula for compact linear operators.

Theorem A.4.4. [see [24, Ch. IV, §12, Thm. 8.3]]
Assume that $L : X \to X$ *is a compact linear operator on a Banach space* X *and that* $1 \notin \sigma(L)$*. Then the set* $\sigma(L) \cap (1, +\infty)$ *consists of a finite number of eigenvalues, each with finite algebraic multiplicity. Moreover, for any open bounded subset* $U \subset X$ *such that* $0 \in U$*, the following formula holds:*

$$\mathrm{Ind}_{LS}(L, U) = (-1)^m$$

where m *is the sum of the algebraic multiplicities of the eigenvalues of* L *belonging to* $(1, +\infty)$*.*

We now present the theory of the topological index for k-set contractions. Let X be a Banach space (not necessarily finite-dimensional), let χ be a Hausdorff measure of non-compactness on X, and let $V \subset X$ be an arbitrary subset – not necessarily open or bounded. A map $\Phi : V \to X$ is said to be a *k-set contraction with respect to* χ if there exists $k \in [0, 1)$ such that, for all bounded sets $W \subset V$,

$$\chi(\Phi(W)) \leq k\chi(W). \tag{A.4}$$

Definition A.4.5. Suppose that X is a Banach space, and that $U \subset X$ is an open bounded subset. We say that a map $\Phi : \overline{U} \to X$ is *admissible for the index for k-set contractions*, if Φ is a continuous k-set contraction and $\Phi(x) \neq x$ for all $x \in \partial U$.

Additionally, a map $H : \overline{U} \times [0, 1] \to X$ is called an *admissible homotopy for the index for k-set contractions* if H is continuous, there exists $k \in [0, 1)$ such that, for any subset $W \subset \overline{U}$,

$$\chi(H(W \times [0, 1])) \leq k\chi(W),$$

and $H(x, \mu) \neq x$ for all $x \in \partial U$ and $\mu \in [0, 1]$.

We briefly recall the construction of the topological index for k-set contractions (see [1, Section 3.1]). Let the map Φ be as before. A non-empty compact convex subset $R \subset X$ is said to be *fundamental* for the map Φ if $\Phi(\overline{U} \cap R) \subset R$ and, for any $x \in \overline{U}$, the condition $x \in \overline{\mathrm{conv}}(\{\Phi(x)\} \cup R)$ implies $x \in R$, where recall that $\overline{\mathrm{conv}}(M)$ means the closed convex hull of a subset $M \subset X$. It can be shown that there exists a fundamental set R for the map Φ such that $\overline{U} \cap R \neq \emptyset$ (see [1, Thm. 3.1.4]). Observe that all fixed-points of the map Φ lie in the set R. Moreover, for any fundamental set R for the map Φ such that $\overline{U} \cap R \neq \emptyset$, there exists a continuous extension $\widetilde{\Phi} : \overline{U} \to X$ of the restriction $\Phi|_{\overline{U} \cap R}$ such that $\widetilde{\Phi}(\overline{U}) \subset R$. Therefore, the map $\widetilde{\Phi}$ is completely continuous. Since $\Phi(x) \neq x$ for $x \in \partial U$, we see that the map $\widetilde{\Phi}$ is admissible for the Leray-Schauder topological index. We can thus define an integer $\mathrm{Ind}_C(\Phi, U)$, called the *topological index for k-set contractions*, as follows:

$$\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_{LS}(\widetilde{\Phi}, U). \tag{A.5}$$

One can prove that this definition above does not depend on the choice of the fundamental set or on the extension $\widetilde{\Phi}$ (see [1, Subsection 3.1.8]).

The following result shows that the index for k-set contractions is independent of the choice of an equivalent norm under which the map remains a k-set contraction.

Lemma A.4.6. *Let* $(X, \|\cdot\|_1)$ *be a Banach space, let* $U \subset X$ *be an open bounded subset, and let the map* $\Phi : \overline{U} \to X$ *be admissible for the index for* k*-set contractions in* $(X, \|\cdot\|_1)$*. If* $\|\cdot\|_2$ *is a norm equivalent to* $\|\cdot\|_1$ *such that* Φ *is a* k*-set contraction in* $(X, \|\cdot\|_2)$ *(possibly with a different constant* k *in inequality* (A.4)*), then*

$$\mathrm{Ind}_{C,1}(\Phi, U) = \mathrm{Ind}_{C,2}(\Phi, U) \tag{A.6}$$

where $\mathrm{Ind}_{C,1}(\Phi, U)$ *denotes the topological index for* k*-set contractions in* $(X, \|\cdot\|_1)$*, and* $\mathrm{Ind}_{C,2}(\Phi, U)$ *denotes the corresponding index in* $(X, \|\cdot\|_2)$*.*

Proof. Let $R \subset X$ be a fundamental set for the map Φ, considered in the space $(X, \|\cdot\|_1)$, and let $\widetilde{\Phi} : \overline{U} \to X$ be a continuous extension of the restriction $\Phi|_{\overline{U}\cap R}$ such that $\widetilde{\Phi}(\overline{U}) \subset R$. By definition (A.5), we have

$$\mathrm{Ind}_{C,1}(\Phi, U) = \mathrm{Ind}_{LS,1}(\widetilde{\Phi}, U) \tag{A.7}$$

where $\mathrm{Ind}_{LS,1}(\cdot,\cdot)$ denotes the Leray-Schauder topological index in $(X, \|\cdot\|_1)$.

By assumption, $\Phi(x) \neq x$ for $x \in \partial U$, and Φ is a k-set contraction in $(X, \|\cdot\|_2)$. Hence, we see that the map Φ is also admissible for the index for k-set contractions in $(X, \|\cdot\|_2)$. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms, it follows that R is also a fundamental set for Φ, considered in $(X, \|\cdot\|_2)$. Similarly, the map $\widetilde{\Phi}$ is a continuous extension of the restriction $\Phi|_{\overline{U}\cap R}$, considered in $(X, \|\cdot\|_2)$, such that $\widetilde{\Phi}(\overline{U}) \subset R$. Therefore,

$$\mathrm{Ind}_{C,2}(\Phi, U) = \mathrm{Ind}_{LS,2}(\widetilde{\Phi}, U) \tag{A.8}$$

where $\mathrm{Ind}_{LS,2}(\cdot,\cdot)$ denotes the Leray-Schauder topological index in $(X, \|\cdot\|_2)$.

Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, the topological invariance of the Leray-Schauder index (see Theorem A.4.3, property (LS8)) yields

$$\mathrm{Ind}_{LS,1}(\Phi, U) = \mathrm{Ind}_{LS,2}(\Phi, U).$$

Combining this with (A.8) and (A.7), we obtain (A.6), which completes the proof. □

The lemma above allows us to slightly extend the notion of the topological index for k-set contractions.

Definition A.4.7. Let $(X, \|\cdot\|)$ be a Banach space, and let $U \subset X$ be an open bounded subset. A map $\Phi : \overline{U} \to X$ is said to be *admissible for the topological index for* k*-set contractions in* $(X, \|\cdot\|)$ if there exists a norm $\|\cdot\|_1$, equivalent to $\|\cdot\|$, such that Φ is admissible for the index for k-set contractions in the space $(X, \|\cdot\|_1)$, in the sense of Definition A.4.5.

We then define the *topological index for* k*-set contractions in* $(X, \|\cdot\|)$ by

$$\mathrm{Ind}_{C}(\Phi, U) = \mathrm{Ind}_{C,1}(\Phi, U) \tag{A.9}$$

where $\mathrm{Ind}_{C,1}(\Phi, U)$ denotes the topological index for k-set contractions in the space $(X, \|\cdot\|_1)$, as defined in (A.5).

Moreover, a map $H : \overline{U} \times [0, 1] \to X$ is said to be an *admissible homotopy for the index for* k*-set contractions in* $(X, \|\cdot\|)$ if there exists a norm $\|\cdot\|_1$, equivalent to $\|\cdot\|$, such that H is an admissible homotopy for the index for k-set contractions in the space $(X, \|\cdot\|_1)$, in the sense of Definition A.4.5.

Theorem A.4.8. [see [1, Subsection 3.1.9]]
For any map $\Phi : \overline{U} \to X$ *that is admissible for the index for* k*-set contractions in the sense of Definition A.4.7, the integer* $\mathrm{Ind}_C(\Phi, U)$*, given by* (A.9)*, satisfies the following properties:*

(C1) *(Normalization) If* $\Phi(x) = x_0$ *for all* $x \in \overline{U}$*, with* $x_0 \in U$*, then* $\mathrm{Ind}_C(\Phi, U) = 1$.

(C2) *(Additivity) If open disjoint sets* U_1, $U_2 \subset U$ *are such that* $\Phi(x) \neq x$ *for* $x \in U \setminus (U_1 \cup U_2)$, *then*

$$\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_C(\Phi|_{\overline{U}_1}, U_1) + \mathrm{Ind}_C(\Phi|_{\overline{U}_2}, U_2).$$

(C3) *(Homotopy invariance) If a map* $H : \overline{U} \times [0,1] \to X$ *is an admissible homotopy for the index for* k*-set contractions, then*

$$\mathrm{Ind}_C(H(\cdot, 0), U) = \mathrm{Ind}_C(H(\cdot, 1), U).$$

(C4) *(Existence) If* $\mathrm{Ind}_C(\Phi, U) \neq 0$, *then there exists* $x \in U$ *such that* $\Phi(x) = x$.

(C5) *(Excision)* [see [21, Thm. 9.2 (d), property (D7)]] *If an open set* $U_1 \subset U$ *is such that* $\Phi(x) \neq x$ *for* $x \in U \setminus U_1$, *then* $\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_C(\Phi|_{\overline{U}_1}, U_1)$.

(C6) *(Contraction)* [see [1, Subsection 3.4.4]] *If* $\Phi(\overline{U}) \subset X_0$, *where* X_0 *is a closed subspace of* X, *then*

$$\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_C\left(\Phi|_{\overline{U} \cap X_0}, U \cap X_0\right).$$

The next result provides a variant of the multiplicativity property of the index.

Theorem A.4.9. [see [21, Thm. 9.3 (b)]]
Suppose that a Banach space X *decomposes as a topological direct sum:* $X = X_1 \oplus X_2$, *and let* $U \subset X$ *be an open bounded set that decomposes as an algebraic sum:* $U = U_1 + U_2$, *where* $U_1 \subset X_1$ *and* $U_2 \subset X_2$ *are open bounded subsets. Assume further that* $\Phi : \overline{U} \to X$ *is a continuous* k*-set contraction such that* $\overline{U}_1$ *and* $\overline{U}_2$ *are invariant under* Φ, *and that*

$$\Phi(x) \neq x \quad \text{for } x \in \partial U, x \in \partial U_1, \text{and } x \in \partial U_2.$$

Then the restricted maps $\Phi|_{\overline{U}_1}$ *and* $\Phi|_{\overline{U}_2}$ *are continuous* k*-set contractions in the spaces* X_1 *and* X_2, *respectively, and the following formula holds:*

$$\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_C(\Phi|_{\overline{U}_1}, U_1) \cdot \mathrm{Ind}_C(\Phi|_{\overline{U}_2}, U_2).$$

Let Φ_0, $\Phi_1 : \overline{U} \to X$ be maps admissible for the index for k-set contractions. We say that Φ_0 and Φ_1 are *homotopic*, and write $\Phi_0 \sim_H \Phi_1$, if there exists an admissible homotopy $H : \overline{U} \times [0,1] \to X$ for the index for k-set contractions such that $H(\cdot, 0) = \Phi_0$ and $H(\cdot, 1) = \Phi_1$. We now present a result on the homotopy classes determined by the relation $\sim_H$.

Theorem A.4.10. [see [1, Subsection 3.1.11]] *The homotopy relation* $\sim_H$ *on the set of all admissible maps* $\Phi : \overline{U} \to X$ *for the index for* k*-set contractions is an equivalence relation. Furthermore, each homotopy class contains a completely continuous map* $\widetilde{\Phi} : \overline{U} \to X$ *such that, for any* $\Phi : \overline{U} \to X$ *belonging to this class,*

$$\mathrm{Ind}_C(\Phi, U) = \mathrm{Ind}_{LS}(\widetilde{\Phi}, U).$$

In particular, if $\Phi : \overline{U} \to X$ *is completely continuous, then its topological index for* k*-set contractions coincides with its Leray-Schauder index.*

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